# BIJECTIVE COMBINATORICS OF REDUCED DECOMPOSITIONS

A Thesis

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# Abstract

We study the bijective combinatorics of reduced words. These are fundamental objects in the study of Coxeter groups. We restrict our focus to reduced words of permutations and signed permutations.

Our results can all be situated within the context of two parallels. The first parallel is between the enumerative theory of reduced words and that of Coxeter group elements. The second parallel is between the combinatorics of permutations and that of signed permutations.

The enumerative properties of reduced words have been studied widely since Stanley first proposed the problem of counting reduced words of permutations. One approach is via the Little map, a bijection introduced by David Little. We introduce a Little map for reduced words of signed permutations and explain how to think of these maps as an analogue jeu de taquin for the setting of reduced words.

We also extend dual equivalence and Rothe diagrams graphs to the setting of signed permutations. In the former case, this leads to a new description of reduced words for signed permutations as balanced labelings of diagrams.

Much of the work introduced in this thesis appeared for the first time in [17], joint with Benjamin Young, and in [6], joint with Sara Billey, Austin Roberts and Benjamin Young.

# Acknowledgements

No work exists in a vacuum. In addition to acknowledging those who have worked on and published the results my work depends on, I would like to single out several those who have helped me personally. I have had the good fortune to have worked with three excellent mentors. My advisor Peter Winkler has been wonderful, giving me extraordinary support while encouraging me to pursue my interests. Benjamin Young taught me how to collaborate, and I have become a better mathematician and person for it. Sara Billey is an inspirational role model, whose perspective has dramatically shifted my thoughts on our work. I hope to live up to the confidence they have shown in me.

The mathematicians who have assisted in this work are too many to name. However, I would be remiss not to mention Austin Roberts, who has taught me most of what I know about dual equivalence graphs. Additionally, the conversations on random sorting networks I have had with Ander Holroyd and Alex Rozinov have had a profound influence on me.

This work could not have happened without the mathematical communities I have been a part of. The Dartmouth mathematics department has supported me in every way imaginable; academically, financially and emotionally. I single out my cohort: Danny, Kassie, Scott, Seth and Zeb. MSRI broadened my perspective, and much of this work began in conversations there.

Finally, I must acknowledge the people who have helped me grow into the person that I am. At the core of this sits my family: my mother, father and three sisters. Amongst my friends, I single out Erica Layton, whose friendship has meant the most to me during my time at Dartmouth.

# Contents

# Chapter 1

# Introduction

## 1.1 Sorting networks

The genesis of this thesis was a software demonstration by Ander Holroyd at an MSRI workshop on Percolation and Interacting Systems. His software depicted random sorting networks. A *swap* is an adjacent transposition $s_i = t_{i,i+1} = (i, i+1)$. Let $N = \binom{n}{2}$ and $w_0 = n(n-1)\dots 1 \in S_n$. An *$n$ element sorting network* is a word $a = a_1 a_2 \dots a_N$ such that

$$s_{a_1} s_{a_2} \dots s_{a_N} = w_0.$$

The set of all sorting networks is denoted $R(w_0)$. In [1], Angel, Holroyd, Romik and Virag initiated the study of uniformly random sorting networks and made several striking conjectures.

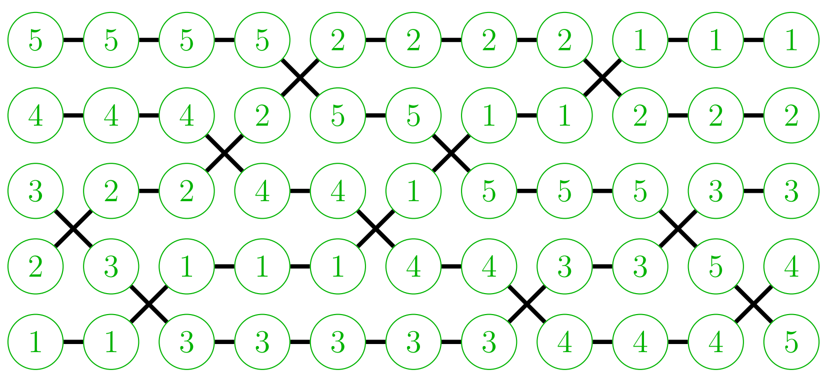
To depict sorting networks, we use *wiring diagrams*, introduced by Goodman and

Figure 1.1: Select trajectories for a uniformly random 2000 element sorting network
Picture of large uniform SN, select trajectories from Holroyd applet

Pollack. For a sorting network $a = a_1 \dots a_N$, let

$$w^k(a) = s_{a_1} \dots s_{a_k} \in S_n$$

where $S_n$ is the group of permutations of $[n] = \{1, 2, \dots, n\}$. The *trajectory* or $j$ in the sorting network is $\{(w^k)^{-1}(j)\}_{k=0}^{N}$. The *wiring diagram* of $a$ is a function $\{0, 1, \dots, N\} \times [n] \to [n]$ such that $((w^k)^{-1}(j), j) \mapsto j$ for all $j \in [n]$ and $k \in \{0, 1, \dots, N\}$. When depicting the wiring diagram, we connect each trajectory via a wire, as shown below for the five element sorting network 2134231421.




Note the $k$th column in the wiring diagram of $a$ is $w^k(a)$ in one line notation. Therefore consecutive columns differ by a single swap.

Holroyd's software, available on his website [20], depicts the wiring diagrams of uniformly random sorting networks (see Figure 1.1). As can be seen, these wiring diagrams have a striking feature. The trajectories appear to converge to sin waves of random phase and amplitude. This is conjectural. The current state of the art is that a scaling limit exists, with certain continuity constraints.

The known results about random sorting networks rely on two key tools. The first is the Edelman-Greene bijection between sorting networks and staircase-shaped standard Young tableaux, due to Edelman and Greene in [9]. This bijection is defined in Section 2.5. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ a partition, a *Young tableaux* of *shape* $\lambda$ is a labeling of $(i, j)$ for $j \in [k]$ and $0 < i < \lambda_{-j}$. Tableaux are typically depicted in diagram form, as shown below with $\lambda = (3, 2, 1)$.

$$\begin{array}{|c|c|c|}
\hline
1 & 3 & 4 \\
\hline
2 & 5 & \multicolumn{1}{c}{} \\
\cline{1-2}
6 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}$$

A tableaux is *standard* if its entries are $[n]$ and they increase across rows and down columns. It is *staircase* if its shape is $\lambda_0^n = (n-1, n-2, \ldots, 1)$. The above tableau is a staircase shaped standard Young tableau. We will suppress the $n$ in $\lambda_0^n$ when its value is clear from context. The second result regards the scaling limit of staircase shaped standard Young tableaux, based on work of Pittel and Romik in [32].

One approach to solving the conjectures about uniformly random sorting networks would be to improve the combinatorial understanding of the Edelman-Greene bijection. In private communication, Holroyd suggested exploring another bijection between sorting networks and staircase shaped standard Young tableau called the the Little map, discussed in detail in Chapter 3. This map, introduced by Little in [27], was conjectured to be the same map as the Edelman-Greene bijection. The initial work in this thesis was directed towards resolving this conjecture. However, the combinatorics of sorting networks belongs to a much larger body of knowledge regarding the combinatorics of Coxeter groups. Our results have so far failed to yield new results about sorting networks, but has led to exciting new developments in the combinatorics of reduced expressions of Coxeter group element.

## 1.2 Reduced expressions and reduced words

A *Coxeter group* $W$ is a group with finite generator set $\{s_1, s_2, \ldots, s_n\}$ and relations

(a) $s_i^2 = 1$.

(b) For $i \neq j$ there exists $m_{ij}$ such that $(s_i s_j)^{m_{ij}} = 1$.

For $w \in W$, the *length* $\ell(w)$ is $\min_q\{s_{a_1} \ldots s_{a_q} = w\}$. We say $s_{a_1} \ldots s_{a_p}$ is a *reduced expression* of $w \in W$ if $s_{a_1} \ldots s_{a_p} = w$ and $\ell(w) = p$. The corresponding word of indices $a_1 \ldots a_p$ is called a *reduced word*, and the set of all reduced words of $w$ is denoted $R(w)$. Reduced expressions are a necessary tool in studying generic Coxeter groups.

We restrict our focus to two families of finite Coxeter groups. The first is the symmetric group $S_{n+1}$, also referred to as the Coxeter group of Type A, with generator set $\{s_1, \ldots, s_n\}$ and relations

(a) $s_i^2 = 1$,

(b) $s_i s_j = s_j s_i$ for $|i - j| > 1$ (equivalently $(s_i s_j)^2 = 1$),

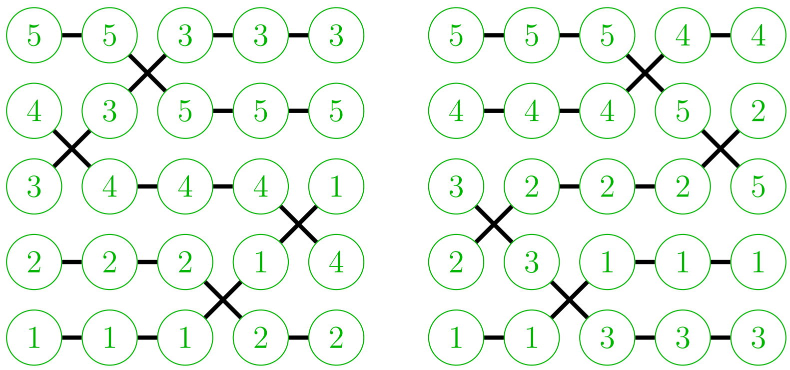
(c) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ (equivalently $(s_i s_{i+1})^3 = 1$).

Here $s_i$ can be thought of as the elementary transposition $t_{i(i+1)} = (i, i{+}1)$. We will often find it convenient to view $S_n$ as lying inside the direct limit $S_\infty$. For example, for $24153 \in S_5$ we have

$$R(24153) = \{3412, 3142, 1342, 3124, 1324\}.$$

Note these are related by commuting Coxeter relations. In general, a theorem of Tits shows that the graph induced on $R(w)$ by the Coxeter relations is connected. Sorting networks are reduced words of the reverse permutation $w_0$.

Wiring diagrams extend naturally to all reduced words of permutations. Every wiring diagram should be considered as a subdiagram of the diagram with wires labeled by all of $\mathbb{N}$ where constant trajectories are suppressed. We depict the wiring diagram of $3412 \in R(24153)$ and $2143 \in R(42513)$ below.



Note 42513 is the inverse of 35142. In general, if $a_1 a_2 \dots a_p \in R(w)$, then $a_p a_{p-1} \dots a_1 \in R(w^{-1})$.

The second group of interest is the hyperoctahedral group $B_n$, referred to as the Coxeter group of Type B. This is the group of signed permutations. Recall $[n] = \{1, \dots, n\}$ and let $[-n] = \{-n, \dots, -1\}$, which we identify with $\{\bar{n}, \dots, \bar{1}\}$. A *signed permutation* is a map $w : [n] \to [-n] \cup [n]$ such that $|w| : [n] \to [n]$ with $i \mapsto |w_i|$ is a permutation. As an example, $w = 3\bar{2}1\bar{4} \in B_4$ where $\bar{2}$ denotes $-2$ and $\bar{4}$ denotes $-4$. The signed permutation $w$ can be extended to a map with domain $[-n] \cup [n]$ by setting $w(-k) = -w(k)$. The previous example would then become $4\bar{1}2\bar{3}3\bar{2}1\bar{4}$. We consider these characterizations to be equivalent, using whichever is more convenient. We will occasionally consider the *flatten operator* fl mapping a string of distinct numbers to the permutation with the same relative order. For example, $\mathrm{fl}(4\bar{1}2\bar{3}3\bar{2}1\bar{4}) = 84627351$. Note the mirror symmetry between the first four and last four values. This group has generator set $\{s_0, s_1, \ldots, s_{n-1}\}$ and relations

(a) $s_i^2 = 1$,

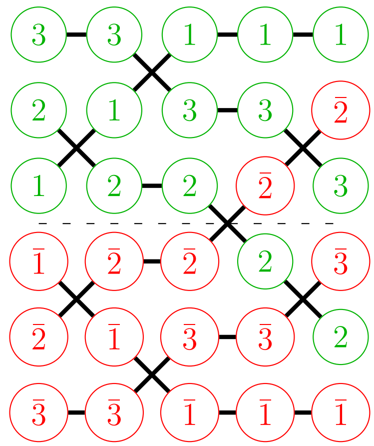
(b) $s_i s_j = s_j s_i$ for $|i-j| > 1$,

(c) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i > 0$,

(d) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

For $w : [n] \to [-n] \cup [n]$ we have $s_i = t_{i,i+1}$ for $i > 0$, while $s_0$ changes the sign of the first position. For $w : [-n] \cup [n] \to [-n] \cup [n]$, the $s_i$ act on positive and negative parts simultaneously as $t_{i,i+1}t_{-i-1,-i}$ while $s_0 = t_{(-1),1} = (-1, 1)$. For example, $R(3\bar{2}1) = \{1021, 1201\}$. Note $S_{n+1} = A_n \subset B_n$. Again, we find it useful to consider signed permutations as residing in the direct limit $B_\infty$.

For $w \in B_n$ and $a = a_1 \ldots a_p \in R(w)$, the *wiring diagram* of $a$ is the function $\{0, 1, \ldots, p\} \times ([-n] \times [n]) \to [-n] \times [n]$ such that $((w^k)^{-1}(j), j) \mapsto j$ for all $j \in [-n] \times [n]$ and $k \in \{0, 1, \ldots, p\}$. If $a_j \neq 0$, there will be two crosses between the $j$th and $j{+}1$th column, while for $a_j = 0$ there will be just one cross between these columns. We connect entries with the same label by "wires" with labels $[-n] = \{-1, -2, \ldots, -n\}$ and $[n] = \{1, 2, \ldots, n\}$ starting on the left hand edge of the diagram. If $a_1 > 0$ then wires $a_1$ and $a_1 + 1$ cross in between columns 0 and 1, as do the wires $-a_1$ and $-a_1 - 1$ cross. If $a_1 = 0$, then wires 1 and $-1$ cross in column $p$. Below, we depict the wiring diagram of 1201.




The length of a permutation or signed permutation has a simple combinatorial interpretation. For a signed permutation $w$, an *inversion* is a pair $(w(j) < w(i))$ with $i < j$. If $(w(j), w(i))$ is an inversion in $w$ then $(i, j)$ is an inversion in $w^{-1}$. Note if $(w(j), w(i))$ is an inversion then $(-w(i), -w(j))$ is as well. The set of inversions is $\mathrm{Inv}(w) = \{(w(j), w(i)) : \ i < j\}/\{(i, j) \sim (-j, -i)\}$. The inversions $(w(j), w(i))$ such that $|w(j)| \leq |w(i)|$ form a set of representatives for $\mathrm{Inv}(w)$. For ordinary permutations, the equivalent definition coincides with the standard definition of an inversion set. A word $b = b_1 \ldots b_n$ has a *descent* at position $i$ if $b(i) > b(i{+}1)$. The descent set of $b$ is $des(b) = \{i \in [n-1] : \ b(i) > b(i{+}1)\}$. Note for $w = w_{\bar{n}} \ldots w_{\bar{1}} w_1 \ldots w_n$ a signed permutation with descent at position $i$ that $-i{-}1$ is also a descent for $i \neq -1$. We present the well-know relationship between length and inversions.

**Lemma 1.1.** *Let $w \in B_\infty$. Then $\ell(w) = |Inv(w)|$.*

*Proof.* Let $w \in B_\infty$ and $a_1 \ldots a_p \in R(w)$. Each $s_{a_i}$ either introduces or removes an inversion. Therefore we see $\ell(w) \geq \mathrm{Inv}(w)$. We show this can be attained by induction.

For $\ell(w) = 1$ we see $w = s_i$ for some $i$. We then see $|\mathrm{Inv}(w)| = 1$. Assume the result holds for $v \in B_\infty$ with $\ell(v) = p$ and let $\mathrm{Inv}(w) = p + 1$. Since $w$ has some inversion, it is not increasing hence must have some descent $i$ with $i \geq -1$. Then $\mathrm{Inv}(ws_i) = p$, so $ws_i$ (or $ws_0$ if $i = -1$) has some reduced expression $s_{a_1} \dots s_{a_p}$. We now see $w = s_{a_1} \dots s_{a_p} s_i$, hence $\ell(w) \leq p + 1$. Thus $\ell(w) = |\mathrm{Inv}(w)|$.

□

Note any inversion introduced by $s_0$ does not appear in $\mathrm{Inv}(w)$ for $w \in S_\infty$, so 0 cannot appear in a reduced word of $w$, even when embedded in $B_\infty$. This is the first of many results that we will prove for Type B that also hold for Type A.

## 1.3 Symmetric functions

The first approach to enumerating reduced words in Type A, introduced in [37], makes use of the theory of symmetric functions. We introduce standard terminology, based loosely on the descriptions in [31] and [38]. A formal power series $f$ in the variables $x_1, x_2, \dots$ is called a *symmetric function* if

$$f(x_1, x_2, \dots) = f(x_{w(1)}, x_{w(2)}, \dots)$$

for any $w \in S_\infty$. The ring of symmetric functions, denoted $\Lambda$, is the set of all symmetric functions with finite degree. It has several bases, indexed by integer partitions. For our purposes the most important are the Schur functions, which we will define in terms of tableaux.

Recall the definition of a tableau from Section 1.1. A tableau is *semistandard* if the labels are weakly increasing across rows and strictly increasing down columns. Let

$SSYT(\lambda)$ denote the set of semistandard Young tableaux of shape $\lambda$. To a tableau $T$ with labels $i_1, \ldots, i_n$ we associate the monomial $x^T = x_{i_1} \ldots x_{i_n}$.

**Definition 1.1.** For a partition $\lambda$, we define the associated *Schur function* by

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^T.$$

A symmetric function $F$ is *Schur positive* if $F = \sum_\lambda c_\lambda s_\lambda$ where $c_\lambda \in \mathbb{N}$ for all $\lambda$.

Schur functions have another characterization in terms of quasisymmetric functions. A formal power series $f$ in the variables $x_1, x_2, \ldots$ is called a *quasisymmetric function* if the coefficients of $x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}$ and $x_1^{\alpha_1} \ldots x_k^{\alpha_k}$ are the same for every sequence $i_1 < \cdots < i_k$. The ring of quasisymmetric functions, denoted QSYM, is the set of all quasisymmetric functions of finite degree. Note $\Lambda \subset$ QSYM. The bases of QSYM are indexed by compositions. Compositions of $n$ are in bijection of elements of $\{-,+\}^{n-1}$, which we call *signatures*.

**Definition 1.2.** For a signature $\sigma = \sigma_1 \ldots \sigma_{n-1} \in \{-,+\}^{n-1}$, we define the *fundamental quasisymmetric function*

$$f_\sigma = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \ldots x_{i_n}$$

where $i_j < i_{j+1}$ for $\sigma_j = -$. Each such sequence $i_1, i_2, \ldots, i_n$ is called a *compatible sequence*, and the set of all compatible sequences of $\sigma$ is denoted $K(\sigma)$.

For example,

$$f_{+-+} = x_1^2 x_2^2 + x_1^2 x_3^2 + \cdots + x_2^2 x_3^2 + x_2^2 x_4^2 + \cdots + x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + \ldots$$

$$+ x_1x_2x_3^2 + x_1x_3x_4^2 + \cdots + x_1x_2x_3x_4 + x_1x_2x_3x_5 + \ldots.$$

Many important symmetric functions can be expressed in terms of fundamental quasisymmetric functions.

Let $T$ be a standard Young tableau of shape $\lambda \vdash n$. We say $T$ has a *descent* at position $i$ if $i$ is in a higher row than $i+1$. The *signature* of $T$, denoted $\sigma_T = \sigma_1 \ldots \sigma_{n-1}$, is defined by setting $\sigma_j = +$ is $T$ has a descent at $j$ and $\sigma_j = -$ otherwise.

**Theorem 1.1** ([14]). *Let $\lambda$ be a partition. Then*

$$s_\lambda = \sum_{T \in SYT(\lambda)} f_{\sigma_T}.$$

A proof can be found in [29]. The standard Young tableaux of shape $\lambda = (2,1,1)$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

with associated descent signatures $-++, +-+$ and $++-$. Then

$$s_{(2,1,1)} = f_{-++} + f_{+-+} + f_{++-}.$$

The initial approach to enumerating reduced words of Type B makes heavy use of subring of a symmetric functions. A symmetric function $f$ *satisfies the $Q$-cancellation condition* if

$$f(x_1, -x_1, x_3, x_4, \ldots) = f(x_3, x_4, \ldots).$$

The ring of such symmetric functions, denoted $\tilde{\Lambda}$, has bases indexed by strict partitions. The most important bases are the $P$-Schur and $Q$-Schur functions. We define them in terms of certain shifted Young tableaux, following [39]. For $\lambda = (\lambda_1 > \cdots > \lambda_k)$ a partition, a *shifted Young tableau* is a labeling of $(i, j)$ for $i \leq j \leq \lambda_{-i}$ with labels in $\mathbb{N}$ and $\bar{\mathbb{N}} = \{\bar{1}, \bar{2}, \ldots\}$. These are best understood as diagrams.

| | | | |
|---|---|---|---|
| 1 | $\bar{2}$ | 2 | 2 |
| | 3 | $\bar{4}$ | |
| | | $\bar{4}$ | |

Such a tableau is $Q$*-semistandard* if it respects the order $\bar{1} < 1 < \bar{2} < 2 < \ldots$, with unbarred labels row strict and barred labels column strict. It is $P$*-semistandard* is the first label in each row is unbarred. Let $|\bar{i}| = i$. The above example is $Q$-semistandard but not $P$-semistandard since the third row begins with a barred label. For a strict partition $\lambda$ let $Q\text{-}SSYT(\lambda)$ and $P\text{-}SSYT(\lambda)$ denote the set of $Q$-semistandard and $P$-semistandard tableaux of shape $\lambda$, respectively. The set of shifted standard Young tableaux of shape $\lambda$ is denoted $ShSYT(\lambda)$.

To a shifted tableau $T$ with labels $i_1, i_2, \ldots, i_n$ we associate the monomial $x^T = x_{|i_1|} \ldots x_{|i_n|}$.

**Definition 1.3.** For $\lambda$ a strict partition, we define the $Q$-Schur function

$$Q_\lambda = \sum_{T \in Q-SSYT(\lambda)} x^T$$

and the $P$-Schur function

$$P_\lambda = \sum_{T \in P-SSYT(\lambda)} x^T.$$

Note $Q_\lambda = 2^{\ell}(\lambda) P_\lambda$.

There are peak quasisymmetric functions, and their relationship with symmetric functions satisfying the $Q$-cancellation property is analogous to the relationship between quasisymmetric functions and symmetric functions. They are indexed by signatures with no consecutive + signs.

**Definition 1.4.** For $\sigma' = \sigma'_2 \dots \sigma'_{n-1}$ a signature with no consecutive + signs, the associated *peak fundamental quasisymmetric function* is

$$\Theta_{\sigma'} = \sum_{i_1 \leq dots \leq i_n} 2^{|\{i_1,\dots,i_n\}|} x_{i_1} \dots x_{i_n}$$

where $i_{k-1} = i_k = i_{k+1}$ implies $\sigma'_k = -$.

There is a natural analogue of Theorem 1.1. We say the shifted standard Young tableau $T$ has a descent at position if $i$ is unsigned and in a strictly lower row than $i{+}1$ or $i{+}1$ is signed and in a weakly lower row than $i$. It has a *peak* at position $i$ if it does not have a descent at $i{-}1$ and does at position $i$. Define $\sigma'_T = \sigma'_2 \dots \sigma'_{n-1}$ by setting $\sigma'_i = +$ if $T$ has a peak at position $i$ and $-$ otherwise.

**Theorem 1.2** ([5])**.**

$$P_\lambda = \sum_{T \in ShSYT(\lambda)} \Theta_{\sigma'_T}.$$

## 1.4 Enumerating reduced words

In [37], Stanley initiated the enumeration of reduced expressions in Type A. To do so, he introduced what are now known as the Stanley symmetric functions. For $a = a_1 \dots a_p$ a word, let $des(a) = \{i \in [p-1] : a_i > a_{i+1}\}$. We associate to $a$ the signature $\sigma^a \in \{-,+\}^{n-1}$ where $\sigma^a_i = +$ if $i \in des(a)$ and $-$ otherwise.

**Definition 1.5.** For $w \in S_\infty$, the *Stanley symmetric function* of Type A is

$$F_w = \sum_{a \in R(w)} f_{\sigma^a}.$$

Note that the $x_1 \dots x_p$ coefficient of $f_\sigma$ is one for all $\sigma$. Therefore the coefficient in $F_w$ or $x_1 \dots x_p$, denoted $[x_1 \dots x_p]F_w$, is $|R(w)|$. Since the Schur functions form a basis of $\Lambda$, for $w \in S_\infty$ with $\ell(w) = p$ we see

$$F_w = \sum_{\lambda \vdash p} A_{\lambda w} s_\lambda.$$

From Definition 1.1, we have $[x_1 \dots x_p]s_\lambda = |SYT(\lambda)|$, which can be computed easily using the hook length formula of Frame, Robinson and Thrall introduced in [12]. If $A_{w\lambda} \in \mathbb{N}$ for all $\lambda$, then $F_w$ is always Schur positive. From this, we could conclude that $R(w)$ can be enumerated by standard Young tableaux. By explicitly computing the $A_{\lambda w}$ we would obtain exact counts of $R(w)$.

In [37], Stanley showed that $F_w$ is a Schur function when $w$ is 2143-avoiding. Such permutations are called *vexillary*. Moreover, Stanley conjectured that $F_w$ is Schur positive. We discuss two proofs of this fact. One is bijective and the other algebraic.

The first bijective proof of this conjecture is due to Edelman and Greene. In [9], they introduce a variant of RSK called Edelman-Greene insertion, sometimes called Coxeter-Knuth insertion. For $w \in S_\infty$, Edelman-Greene insertion associates to each $a = a_1 \dots a_p \in R(w)$ a pair of tableaux $(P(a), Q(a))$ where $P(a)$ is row-and-column strict and $Q(a)$ is standard. Let $\mathrm{EG}(w) = \{P(a) : a \in R(w)\}$. For $P \in \mathrm{EG}(w)$ of shape $\lambda$ and $Q \in SYT(\lambda)$, there exists some $a \in R(w)$ such that $P(a) = P$ and

$Q(a) = Q$. From this, Edelman and Greene derive the following theorem.

**Theorem 1.3** ([9]). *For $w \in S_\infty$, we have*

$$A_{\lambda w} = |EG(w) \cap SSYT(\lambda)|.$$

*Therefore $F_w$ is Schur positive.*

We define Edelman-Greene insertion in Chapter 2.

The second approach is based on the theory of Schubert polynomials. In [26], Lascoux and Schützenberger introduced several recurrences on Schubert polynomials. For $w \in S_n$, let $(r, s)$ be the lexicographically largest inversion of $w^{-1}$. Note $\ell(wt_{rs}) = \ell(w) - 1$ and define

$$T(w) = \{wt_{rs}t_{ir} : \ell(wt_{rs}t_{ir}) = \ell(w)\}.$$

If $T(w)$ would be empty, instead set $T(w) = T(1 \oplus w)$ where $1 \oplus w = 1(w_1+1)\dots(w_n+1)$.

**Theorem 1.4.** *For $w \in S_\infty$,*

$$F_w = \sum_{w' \in T(w)} F_{w'}.$$

For $w = 24153$, we see $T(w) = \{34125, 24315\}$.

We say a permutation $w$ is *Grassmannian* if it has exactly one descent. For example, 34125 is Grassmannian. For any $w \in S_\infty$, repeated application of the transition equations results in an expression of $F_w$ as the sum of several $F_{w'}$ where the $w'$ are Grassmannian. In the previous example, $T(24315) = \{235146\}$, which is also Grassmannian. Since Grassmannian permutations are vexillary, the $F_{w'}$ are

Schur functions. We may then use this expression of $F_w$ to compute $|R(w)|$. From this, we see

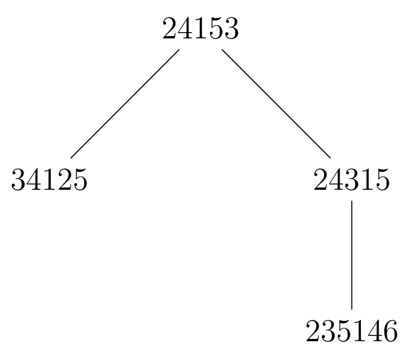
$$F_{24153} = F_{34125} + F_{235146} = s_{(2,2)} + s_{(3,1)},$$

so

$$|R(24153)| = |R(34125)| + |R(235146)| = f^{(2,2)} + f^{(3,1)} = 2 + 3 = 5,$$

as were previously shown.

The enumeration of reduced words via transition equations can be visualized as a tree. If $w \in S_n$ is Grassmannian, it is a leaf. For $w \in S_n$ that is not Grassmannian, the children of $w$ are the permutations in $T(w)$. The *Lascoux-Schützenberger tree* of a permutation $w$ is the tree with root $w$ resulting from these rules. Then $F_w$ is equal to the sum of the Schur functions associated to the leaves in its Lascoux-Schützenberger tree. For example, the Lascoux-Schüzenberger tree of 24153 is




In [27], David Little introduced a bijective realization of the Lascoux-Schützenberger tree. His bijection, called the Little map, acts on a reduced word $a$ by a sequence of operations called Little bumps. Certain canonical Little bumps respect the transition equations, and repeated application of these canonical Little bumps converts $a$ into a reduced word of some Grassmannian permutation. There is a simple bijection between reduced words of Grassmannian permutations and standard Young tableaux of the corresponding shape. Little bumps are invertible, so by retracing the path $a$ follows down the Lascoux-Schützenberger tree the Little map can be reversed.

These approaches to enumeration have natural analogues in the setting of signed permutations. The Type B Stanley symmetric functions were introduced independently in [4] and [11].

**Definition 1.6.** Let $w \in B_\infty$. The *Type C Stanley symmetric function* of $w$ is

$$G_w = \sum_{a \in R(w)} \Theta_{\sigma_a}.$$

The *Type B Stanley symmetric function* is $2^{-\ell_0(w)} G_w$ where $\ell_0(w)$ is the number of values in $[n]$ mapped to negative numbers.

Let $w \in B_\infty$ with $\ell(w) = p$. Note the $x_1 \dots x_p$ coefficient of a peak fundamental quasisymmetric function is $2^p$ for arbitrary signature. Therefore we see $[x_1 \dots x_p]G_w = 2^p|R(w)|$. Moreover, since $G_w$ is projective, it can be expressed in terms of $Q$-Schur functions. For a strict partition $\lambda \vdash n$, from Definition 1.3 we see $[x_1 \dots x_n]Q_\lambda = 2^n|ShSYT(\lambda)|$ (each entry could be barred or unbarred). Therefore $G_w = \sum_\lambda C_{\lambda w} Q_\lambda$. Showing the $C_{\lambda w} \in \mathbb{N}$ would imply $G_w$ is $Q$-Schur positive and computing them explicitly allows for the enumeration of $R(w)$ in terms of shifted standard Young tableaux, which are easily counted.

The first proof that $G_w$ are $Q$-Schur positive for $w \in B_\infty$ is based on an insertion algorithm. We say a shifted tableau is *unimodal* if each row has no peak and columns are strictly decreasing. Introduced in [22], Kraśkiewicz insertion maps $a \in R(w)$ to a pair of tableaux $(P'(a), Q'(a))$ where $P'(a)$ is unimodal and $Q'(a)$ is standard. Let $\mathrm{Kr}(w) = \{P'(a) : a \in R(w)\}$. For $P' \in \mathrm{Kr}(w)$ of shifted shape $\lambda$ and $Q \in ShSYT(\lambda)$

there exists some $a \in R(w)$ such that $P'(a) = P'$ and $Q'(a) = Q'$.

**Theorem 1.5** ([22]). *For $w \in B_\infty$, we see $B_{\lambda w}$ is the number of elements in $Kr(w)$ of shape $\lambda$. Therefore $G_w$ is $Q$-Schur positive.*

A second approach based on transition equations computes the explicit $Q$-Schur expansion of $G_w$. In [4] Billey demonstrates some recurrences on Type B Schubert polynomials and extends them to transition equations on Type B and C Stanley symmetric functions. For a permutation $w$, recall the definition of $T(w)$. When $w$ is a signed permutation, we see $T(w)$ is always non-empty.

**Theorem 1.6.** *For $w \in B_\infty$,*

$$G_w = \sum_{w' \in T(w)} G_{w'}.$$

We say a signed permutation $w$ is *isotropic Grassmannian* if the image of $[n]$ is increasing. Equivalently, $w$ has exactly one descent as $w_{-1} > w_1$ but all other positions are ascents. When $w$ an isotropic Grassmannian permutation we have $G_w = Q_\lambda$ for some strict partition $\lambda$ determined by $w$. Repeated application of the transition equations results in isotropic Grassmannian permutations. In an approach analogous to that of the Lascoux-Schützenberger tree we can then compute $|R(w)|$ for any $w \in B_\infty$.

## 1.5 Two parallels

Every result in this thesis can be viewed as exhibiting one of two types of combinatorial parallels in this thesis. The first is that between the combinatorial tools used to study ordinary and projective representations of the symmetric group. Here, the fundamental objects are Young tableaux, whose shapes are determined by partitions. For representations, the tableaux are not shifted, while in the projective case they are shifted. Associated to these are the Schur functions and $Q$-Schur functions, respectively. These functions form bases for the ring of symmetric functions and the ring of symmetric functions satisfying the $Q$-cancellation property. The key tool connecting tableaux and permutations is the Robinson-Schensted-Knuth algorithm (RSK) and its shifted analogue, the Sagan-Worley algorithm (shifted RSK), defined in Chapter 2. The maps can be interpreted both as insertion algorithms or through the lens of jeu de taquin.

The second type of parallel structure comes in relating these combinatorial tools to the enumerative study of reduced words. The natural analogue of tableaux are the Rothe diagrams of permutations. The clear analogue of Schur functions are the Stanley symmetric functions, which exist for all finite Coxeter group. There are insertion algorithms for reduced words of types $A$, $B$, $C$ and $D$ Coxeter groups that are variants of RSK and shifted RSK. They are due to Edelman-Greene [9], Kraśkiewicz [22] and Tao Kai Lam [23]. In this chapter, we introduced reduced expressions of permutations and signed permutations, develop the basics of symmetric functions and outline the enumerative history of reduced expressions. In Chapter 2, we present the insertion algorithms in greater detail, along with a few novel related results.

One major piece whose role in this picture was not known was the Little map. Little had conjectured a relationship between the Edelman-Greene insertion algorithm and his map in special cases. One of the main results of this thesis is to bring the

Little map into this overall picture. In Chapter 3, we develop an analogue of the Little map for reduced expressions of signed permutations. In Chapter 4, we show that the Little map can be thought of as analogous to the jeu de taquin characterization of RSK. Moreover, the steps in the Little map, called Little bumps, can be thought of as the jeu de taquin slides. These results appear in [17] for the original Little map and in [6] for signed permutations. A candidate for shifted Rothe diagrams is introduced in Chapter 6.

One more piece to introduce into the picture is the theory of dual equivalence graphs, introduced and characterized by Sami Assaf in [3]. In Chapter 2, we show that the obvious extension in the setting of reduced words, called Coxeter-Knuth graphs, are instances of instances of dual equivalence graphs. This result, first observed by Billey in private communications, follows quickly from the theory of dual equivalence. In Chapter 5, we define shifted dual equivalence graphs in the obvious way. This is an instance of the first type of parallel structure. Additionally, we characterize these graphs following Roberts's refinement in [34] of Assaf's characterization. These results appear in [6].

# Chapter 2

# Insertion Algorithms

In this chapter, we define Edelman-Greene insertion and Kraśkiewicz insertion. These are instances of insertion algorithms, the most famous of which is the Robinson-Schensted-Knuth (RSK) algorithm. Generically, insertion algorithms have an insertion rule for inserting a number $k$ into a list of numbers $R$ bumping some output $k'$ (which may be empty). This insertion rule is applied sequentially to the rows of a tableau, with the outputs inserted a row lower until the output is empty. Given a word $k$ and a tableau $T$, the result of inserting $k$ into $T$ is a new tableau $T'$. By inserting the letters of a word sequentially, the insertion algorithm builds a tableau.

Many insertion algorithms have local transformations on the input word that preserve certain properties. Such transformations are a key technique in the study of insertion algorithms. For RSK, these are the Knuth moves. We discuss Coxeter-Knuth moves, which are analogues of Knuth moves for Edelman-Greene and Kraśkiewicz insertion. In order to understand the action of Coxeter-Knuth moves on recording tableaux, we will introduce the notion of dual equivalence, following the treatment of [16].

In general, insertion algorithms are invertible. For Edelman-Greene insertion and Kraśkiewicz insertion, there are special cases where the inverse map has a particularly beautiful characterization based on the Schützenberger's promotion operator. These inverse maps are a key tool in the probabilistic study of certain reduced words called *sorting networks*. We define these maps and introduce new bijections between sorting networks and tableaux.

## 2.1 Edelman-Greene insertion

Edelman-Greene insertion is a variant of the Robinson-Schensted-Knuth (RSK) algorithm that maps reduced words of permutations to pairs of Young tableaux. In order to define this map, we must first define a rule for inserting a number into a tableau. Let $k \in \mathbb{Z}$ and $R = r_1 < r_2 < \cdots < r_l$ be increasing. We define the insertion rule for Edelman-Greene insertion, following [9].

(a) If $n \geq r_l$ or if $R$ is empty, adjoin $n$ to the end of $R$.

(b) If $n < r_l$, let $j$ be the smallest number such that $n < r_j$.

   (i) If $r_j = n + 1$ and $r_{j-1} = n$, set $k' = n + 1$ and leave $R$ unchanged.

   (ii) Otherwise, replace $r_j$ with $k$ and set $k' = r_j$. We say $k$ *bumps* $r_j$.

Edelman-Greene proceeds as a typical insertion algorithm, with the caveat that entries are inserted from right to left. We insert $k$ into a tableau $T$ with increasing rows $R_1, \ldots, R_l$ by inserting $k$ into $R_1$, insert any resulting output $k'$ into $R_2$ and so on until there is the output is empty. Aside from (a)(ii), this is the RSK insertion rule, denoted $(P_{\text{RSK}}, Q_{\text{RSK}})$. Such exceptional bumps are referred to as *special*. For

$a = a_1 \dots a_p$ a word (not necessarily reduced), we define $\mathrm{EG}(a) = (P(a), Q(a))$ via the following sequence of tableaux. We obtain $P_1(a)$ by inserting $a_p$ into the empty tableau. Then $P_j(a)$ is obtained by inserting $a_{p-j+1}$ into $P_{j-1}(a)$. At each step, one additional box is added. In $Q(a)$, the entry of each box records the time of the step in which it was added. From this, we can conclude that $Q(a)$ is a standard Young tableau. For $w \in S_\infty$ and $a \in R(w)$, it is shown in [9] that the entries of $P(a)$ are strictly increasing across rows and down columns. Additionally, we can recover $w$ from $P(a)$ with no additional information, that is $P(a)$ determines $w$. To do so, we take the *reverse row reading word* of $P(a)$, which is the word whose reverse is the rows of $P(a)$ read from bottom to top.

*Example* 2.1. For $a = 32314524 \in R(431652)$, we perform Edelman-Greene insertion.

$$P_1(a) \quad \begin{array}{|c|}\hline 4 \\ \hline\end{array} \qquad\qquad Q_1(a) \quad \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

$$P_2(a) \quad \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline\end{array} \qquad\qquad Q_2(a) \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$$

$$P_3(a) \quad \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad\qquad Q_3(a) \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$P_4(a) \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 4 & 5 \\ \hline\end{array} \qquad\qquad Q_4(a) \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

$$
\begin{array}{c}
P_5(a)\\
\begin{array}{|c|c|}\hline 1 & 4\\\hline 2 & 5\\\hline 4 \\\cline{1-1}\end{array}
\end{array}
\qquad
\begin{array}{c}
Q_5(a)\\
\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & 4\\\hline 5 \\\cline{1-1}\end{array}
\end{array}
$$

$$
\begin{array}{c}
P_6(a)\\
\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & 4\\\hline 4 & 5\\\hline\end{array}
\end{array}
\qquad
\begin{array}{c}
Q_6(a)\\
\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & 4\\\hline 5 & 6\\\hline\end{array}
\end{array}
$$

$$
\begin{array}{c}
P_7(a)\\
\begin{array}{|c|c|}\hline 1 & 2\\\hline 2 & 3\\\hline 4 & 5\\\hline 5 \\\cline{1-1}\end{array}
\end{array}
\qquad
\begin{array}{c}
Q_7(a)\\
\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & 4\\\hline 5 & 6\\\hline 7 \\\cline{1-1}\end{array}
\end{array}
$$

$$
\begin{array}{c}
P_8(a)\\
\begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline 2 & 3 \\\cline{1-2} 4 & 5 \\\cline{1-2} 5 \\\cline{1-1}\end{array}
\end{array}
\qquad
\begin{array}{c}
Q_8(a)\\
\begin{array}{|c|c|c|}\hline 1 & 3 & 8\\\hline 2 & 4 \\\cline{1-2} 5 & 6 \\\cline{1-2} 7 \\\cline{1-1}\end{array}
\end{array}
$$

Here $P(a) = P_8(a)$ and $Q(a) = Q_8(a)$. Note $P$ is row and column strict, while $Q$ is standard. The reverse row reading word of $P(a)$ is $32132545 \in R(431652)$.

Edelman-Greene insertion behaves well with respect to the descents of a reduced word. Recall for $a = a_1 \dots a_p$ that $des(a) = \{i \in [p-1] : a_i > a_{i+1}\}$. Similarly, recall for $T$ a standard Young tableau $i \in des(T)$ if the label $i$ appears in a higher row that the label $i{+}1$.

**Theorem 2.1** (Theorem 6.27 in [9]). *Let $a$ be a reduced word. Then $des(a) = des(Q(a))$.*

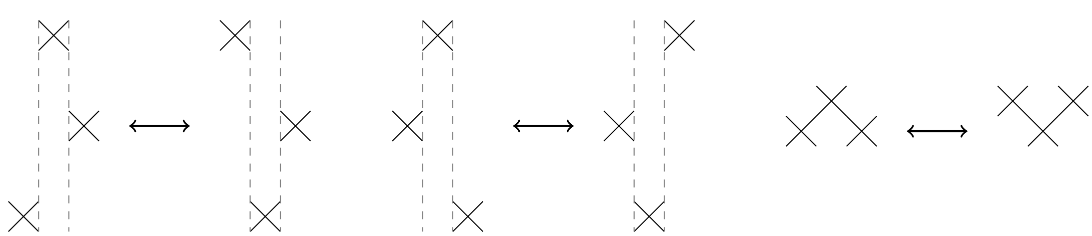
Coxeter-Knuth moves are perhaps the most important tool for studying Edelman-Greene insertion. They are modifications of the second and third Coxeter relations. For $a = a_1a_2a_3$ a word with no consecutive entries the same, let $\alpha$ act on $a$ as follows. For $x < y < z$ integers, Then

(a) $zxy \leftrightarrow xzy$

(b) $yxz \leftrightarrow yzx$

(c) $(x{+}1)x(x{+}1) \leftrightarrow x(x{+}1)x$

while $xyz$ and $zyx$ are fixed. For $a = a_1 \ldots a_p$ a reduced word the *Coxeter-Knuth move* $\alpha_i$ fixes $a_j$ for $j \notin \{i, i{+}1, i{+}2\}$ and acts of $ia_{i+1}a_{i+2}$ by applying $\alpha$. If $\alpha_i$ acts on $a$ non-trivially, we say $a$ *admits* a Coxeter-Knuth move at position $i$. Note $\alpha_i$ is an involution. We depict the three types of Coxeter-Knuth moves, viewed as acting on wiring diagrams:



Type (a) Type (b) Type (c).

We will make heavy use of relationship between Coxeter-Knuth moves and the descent set of a word. Recall for $a = a_1 \ldots a_p$ a reduced word that $des(a) = \{i \in [n-1] : a_i > a_{i+1}\}$. Assume $i \in des(a)$ and $i{+}1 \notin des(a)$. Then $i \notin des(\alpha_i a)$

and $i+1 \in des(\alpha_i a)$. Indeed, this characterizes the presence of a non-trivial Coxeter-Knuth move.

Note the first two types of Coxeter-Knuth moves are refinements of the commuting Coxeter relation while the third corresponds to the braid relation. We then see for $w \in S_\infty$ and $a = a_1 \dots a_p \in R(w)$ that $\alpha_i(a) \in R(w)$ as well. For a word $a' = a_i a_{i+1} a_{i+2}$ on the left $i \in des(a')$ and $i+1 \notin des(a')$ while the reverse holds if $a'$ is on the right hand side. Moreover, all such words are acted on non-trivially by $\alpha$. We say two reduced words $a$ and $b$ are *Coxeter-Knuth equivalent* if there exists a sequence $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}$ of Coxeter-Knuth moves such that

$$b = \alpha_{i_1} \dots \alpha_{i_k} a.$$

Coxeter-Knuth moves play a role in the study of Edelman-Greene insertion analogous to that of Knuth moves in the study of RSK insertion.

**Theorem 2.2** (Theorem 6.24 in [9])**.** *Let $a$ and $b$ be reduced words. Then $P(a) = P(b)$ if and only if $a$ and $b$ are Coxeter-Knuth equivalent.*

Because they span the space of all words with the same insertion tableaux, many statements about Edelman-Greene insertion can be verified by establishing the result for a canonical form and verifying the assertion is invariant under Coxeter-Knuth moves. This will be the strategy for many of the proofs in Chapter 4.

## 2.2 Kraśkiewicz insertion

Kraśkiewicz insertion is a variant of mixed shifted insertion that maps reduced words of signed permutations to pairs of shifted tableaux. In order to define it, we must first define the rule for inserting a number into a tableau. Our presentation is a slight modification of the algorithm in [23], which is equivalent to the algorithm in [22]. Let $k \in \mathbb{N}$ and $R = r_1 > \dots r_{j-1} > r_j < r_{j+1} < \cdots < r_l$.. We first define the insertion of $k$ into $R$.

(a) If $k \neq 0$ or $r_j \neq 0$, perform Edelman-Greene insertion of $k$ into the string $r_{j+1} \dots r_l$. Call the bumped entry $k^-$, if it exists. Call the resulting string after insertion $v_1 \dots v_q$.

(b) If $k = 0$ and $r_j = 0$, set $k^- = 1$. Set $v_1 \dots v_q = r_{j+1} \dots r_l$.

(c) If $k^-$ exists, perform Edelman-Greene insertion of $-k^-$ into $-r_1 \cdots - r_j$. This output will exist, as $k^- > r_j$. Call the bumped entry $k'$. Set $u_1 \dots u_p$ to be the result of negating every entry in the resulting string after insertion and reversing it.

(d) If $k^-$ does not exist, set $u_1 \dots u_p = r_1 \dots r_j$.

We are then left with a new row $R' = u_1 \dots u_p v_1 \dots v_q$ that is unimodal and, if an entry is bumped, a bumped value $k'$. To insert $k$ into a unimodal shifted tableau $T = R_1 \dots R_m$, we insert $k$ into $R_1$, the resulting output $k'$ into $R_2$ and so on until no output $k'$ exists. Given $w \in B_\infty$ and a reduced word $a = a_1 \dots a_p \in R(w)$, we build a sequence of shifted Young tableau $P_1'(a), P_2'(a), \dots P_p'(a) = P'(a)$ by inserting the entries of $a$ from right to left. The recording tableaux $Q_1'(a), \dots, Q_p'(a) = Q'(a)$ are obtained by recording the time step in which each box is added. In [23] it is shown that $P'(a)$ is unimodal and $Q'(a)$ is standard. Additionally, we can recover $w$ from $P'(a)$.

## 2.2 Kraśkiewicz insertion

*Example* 2.2. We perform the Kraśkiewicz insertion of $a = 03421031 \in R(\bar{1}\bar{4}523)$:

$$P'_1(a) \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad\qquad Q'_1(a) \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

$$P'_2(a) \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} \qquad\qquad Q'_2(a) \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

$$P'_3(a) \qquad \begin{array}{|c|c|}\hline 3 & 0 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array} \qquad\qquad Q'_3(a) \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}$$

$$P'_4(a) \qquad \begin{array}{|c|c|c|}\hline 3 & 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array} \qquad\qquad Q'_4(a) \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$$

$$P'_5(a) \qquad \begin{array}{|c|c|c|c|}\hline 3 & 0 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array} \qquad\qquad Q'_5(a) \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$$

$$P'_6(a) \qquad \begin{array}{|c|c|c|c|c|}\hline 3 & 0 & 1 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array} \qquad\qquad Q'_6(a) \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 6 \\ \hline \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$$

$P'_7(a)$

| | | | | |
|---|---|---|---|---|
| 4 | 0 | 1 | 2 | 3 |
| | 1 | 3 | | |

$Q'_7(a)$

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 4 | 5 | 6 |
| | 3 | 7 | | |

$P'_8(a)$

| | | | | |
|---|---|---|---|---|
| 4 | 1 | 0 | 2 | 3 |
| | 3 | 0 | | |
| | | 1 | | |

$Q'_8(a)$

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 4 | 5 | 6 |
| | 3 | 7 | | |
| | | 8 | | |

Here $P'(a) = P'_8(a)$ and $Q'(a) = Q'_8(a)$. We observe that $P'(a)$ has unimodal rows and $Q'(a)$ is standard. Note the last insertion exhibits the exceptional case where $r_j$ and $a_i$ are both zero.

In much the same way Edelman-Greene insertion respects descents, Kraśkiewicz insertion respects the peaks of a reduced word. Recall for $a = a_1 \dots a_p$ a word that $peaks(a) = \{i \in [p-2] : a_i < a_{i+1} > a_{i+2}\}$. Similarly, we say $i$ is a peak in a tableau $T$ if $i{+}1$ is a descent in $T$ but $i$ is not and let $peaks(T)$ denote its peak set.

**Theorem 2.3** (Theorem 2.10 in [23]). *Let $a$ be a Type B reduced word. Then $peaks(a) = peaks(Q'(a))$.*

There are analogues of Coxeter-Knuth moves for Kraśkiewicz insertion. Again, they certain instances of the Coxeter relations. For $a = a_1a_2a_3a_4$ a word with no consecutive entries the same, let $\beta$ act on $a$ as follows. Let $x < y < z < t$. Then

(a) $0101 \leftrightarrow 1010$

(b) $xy(y{+}1)y \leftrightarrow x(y{+}1)y(y{+}1)$

(c) $yx(y{+}1)y \leftrightarrow y(y{+}1)xy$

(d) $x(x+1)yx \leftrightarrow xy(x+1)x$ for $x+1 < y$

(e) $(x+1)xy(x+1) \leftrightarrow (x+1)yx(x+1)$ for $x+1 < y$

(f) $xtyz \leftrightarrow xytz$

(g) $xzyt \leftrightarrow xzty$

(h) $xtzy \leftrightarrow txzy$

(i) $ytxz \leftrightarrow yxtz$

or the reverse of any of these relations. Otherwise, $\beta$ fixes $a$. For $a = a_1 \dots a_p$ the reduced word of a signed permutation, the *Type B Coxeter-Knuth move* $\beta_i$ fixes $a_j$ for $j \notin \{i, i{+}1, i{+}2, i+3\}$ and applies $\beta$ to $a_i a_{i+1} a_{i+2} a_{i+3}$. Note (a) is the long braid relation, (b) is the short braid relation and the remaining non-trivial Type B Coxeter-Knuth relations act via the commuting Coxeter relation. Each of the words on the left has a peak in position $i{+}1$, while on the right the words have a peak in position $i{+}2$. In fact, all reduced words of length four with a peak in the second or third position are acted on non-trivially by $\beta_1$. We say two reduced words $a$ and $b$ are *(Type B) Coxeter-Knuth equivalent* if there exists a sequence of Type B Coxeter-Knuth moves $\beta_{i_1} \dots \beta_{i_k}$ such that

$$b = \beta_{i_1} \dots \beta_{i_k} a.$$

**Theorem 2.4** (Theorem 1.18 in [23]). *Let $a$ and $b$ be Type B reduced words. Then $P'(a) = P'(b)$ if and only if they are Coxeter-Knuth equivalent.*

## 2.3 Jeu de taquin and dual equivalence

Dual equivalence operators were introduced by Haiman in [16] to formalize the relationship between RSK and jeu de taquin on tableaux. They are transformations on standard skew-shaped tableau that, when interpreted as acting on the reading word of the tableau, are dual to Knuth transformations. We must first introduce jeu de taquin.

A skew shape $\lambda$, potentially shifted, can be viewed as lying in a an interval in the posets $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \geq \mathbb{Z}$, respectively. Here, $\mathbb{Z} \geq \mathbb{Z} = \{(i,j) \in \mathbb{Z}^2 : i \geq j\}$ and the partial order is $(i,j) \leq (i',j')$ if $i \leq i'$ and $j \geq j'$. We say a shape $\mu$ *extends* the shape $\lambda$ if $\lambda \cup \mu$ is a skew shape and $\mu$ is after $\lambda$. Let $T$ be a standard tableau. If $\{c\}$ extends $sh(T)$, the *forward slide* on $T$ into the cell $c$ moves the greater of the entries immediately above and to the left of $c$ into $c$. Continue this process until the resulting object is a tableau. The last cell with an entry removed is said to be *vacated*. If $sh(T)$ extends $\{c\}$, the opposite process is a *reverse slide* on $T$ into $c$. Two tableaux are said to be *jeu de taquin equivalent* if they differ by a sequence of forward and reverse slides.

*Example* 2.3. We demonstrate an example of a forward slide.

$$\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 6 & c \\ 5 & & \end{array} \mapsto \begin{array}{ccc} 1 & 2 & 4 \\ 3 & \; & 6 \\ 5 & & \end{array} \mapsto \begin{array}{ccc} 1 & 2 & 4 \\ \; & 3 & 6 \\ 5 & & \end{array} \mapsto \begin{array}{ccc} & 2 & 4 \\ 1 & 3 & 6 \\ 5 & & \end{array}$$

Note that certain slides are only possible for shifted tableau.

$$\begin{array}{ccc} c & 1 & 3 \\ & 2 & \end{array} \mapsto \begin{array}{ccc} 1 & \; & 3 \\ & 2 & \end{array} \mapsto \begin{array}{ccc} 1 & 2 & 3 \end{array}$$

while

$$\begin{array}{ccc} c & 1 & 2 \\ & 3 & \end{array} \mapsto \begin{array}{ccc} 1 & & 3 \\ & 2 & \end{array} \mapsto \begin{array}{ccc} & 1 & 2 \\ & & 3 \end{array}.$$

Here $\{c\} \cup \lambda$ is not a skew diagram, but is a shifted diagram.

Using jeu de taquin slides, we define the *Schützenberger promotion operator* $\Delta$. Given a standard Young tableau $T$ of normal shape $\lambda \vdash n$, we obtain $\Delta(T)$ by:

(a) removing then box labeled 1,

(b) performing a jeu de taquin slide into the newly empty position,

(c) subtracting one each every entry,

(d) adding the label $n$ to the vacated box.

Note $\Delta(T)$ is also a standard Young tableau of shape $\lambda$. Occasionally we will find it convenient to suppress the fourth steps, in which case we write $\Delta'$ instead. The inverse of $\Delta$ is defined similarly, instead starting with the box with the largest label. Continuing Example 2.3, we see

$$\Delta\left(\begin{array}{cc} 1 & 2 \\ 3 & \end{array}\right) = \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$$

and

$$\Delta^{-1}\left(\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 6 & \\ 5 & & \end{array}\right) = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & & \end{array}.$$

The promotion operator can be used to invert insertion algorithms.

**Theorem 2.5.** *(a) [9, Lemma 7.20] Let $w \in S_\infty$ and $a = a_1 \dots a_p \in R(w)$. Then*

$$Q(a_2 \dots a_p) = \Delta'(Q(a_1 \dots a_p)).$$

*(b) [23, Theorem 3.24] Let $w \in B_\infty$ and $a = a_1 \dots a_p \in R(w)$. Then*

$$Q'(a_2 \dots a_p) = \Delta'(Q'(a_1 \dots a_p)).$$

Dual equivalence is defined using jeu de taquin slides.

**Definition 2.1.** Two tableaux $T$ and $U$ are *dual equivalent* if for any sequence of eligible sequence of slides $(c_1, \dots, c_m)$ applied to $T$ and $U$ the resulting shapes are the same.

In [16] Haiman introduced dual equivalence on tableaux and gave several alternative characterizations. These results unify a vast body of folklore on the relationship of jeu de taquin and RSK.

**Theorem 2.6** (Theorem 2.12 in [16])**.** *Two tableaux are dual equivalent if and only if the insertion tableaux of their reading words under RSK are the same. For shifted tableaux, the analogous result holds with respect to Sagan-Worley insertion.*

Haiman also characterized dual equivalence in terms of local transformations on tableaux.

**Definition 2.2.** Let $h$ act on $w \in S_3$ by switching $x$ and $y$ when $w$ is of the form

$$x1y, \ x3y$$

and fixing $w$ otherwise. The *elementary dual equivalence* $h_i$ acts on a standard tableau $T$ (unshifted) by fixing $j$ for $j \notin \{i, i{+}1, i{+}2\}$ and applying $h$ to the reading word of $T\,|_{\{i,i+1,i+2\}}$.

Similarly, let $h'$ act on $w \in S_4$ by switching $x$ and $y$ when $w$ is of the form

$$1x2y,\ x12,\ 1x4y,\ x14y,\ 4x1y,\ x41y,\ 4x3y,\ x43y$$

and fixing $w$ otherwise. The *elementary (shifted) dual equivalence* $h'_i$ acts on a standard shifted tableau $T$ by fixing $j$ for $j \notin \{i, i{+}1, i{+}2, i+3\}$ and applying $h$ to the reading word of $T\,|_{\{i,i+1,i+2,i+3\}}$.

**Theorem 2.7** (Theorem 2.6 in [16])**.** *Two tableaux are dual equivalent if and only if they differ by a sequence of elementary dual equivalences.*

**Proposition 2.1** ([16] Prop 2.14)**.** *For $\lambda$ a normal shape, shifted or unshifted, all standard Young tableaux of shape $\lambda$ are dual equivalent.*

Dual equivalence characterizes the action of Coxeter-Knuth moves on recording tableaux.

**Proposition 2.2.** *(a) Let $w \in S_\infty$ and $a = a_1 \ldots a_p \in R(w)$. Then for $i \in [p-2]$*

$$Q(\alpha_i(a)) = h_i(Q(a)).$$

*(b) Let $w \in B_\infty$ and $a = a_1 \ldots a_p \in R(w)$. Then for $i \in [p-3]$*

$$Q'(\beta_i(a)) = h'_i(Q'(a)).$$

*Proof.* We prove (a) first. Let $w \in S_\infty$ and $a = a_1 \ldots a_p \in R(w)$ with $\alpha_i(a) \neq a$. Since Coxeter-Knuth moves preserve insertion tableaux, we see $Q(a)$ and $Q(\alpha_i(a))$ agree for labels $k \notin \{i, i{+}1, i{+}2\}$. This implies $Q(a)$ and $Q(\alpha_i(a))$ differ by some rearrangement of the labels $i, i{+}1, i{+}2$.

By omitting any extra values at the end of $a$, we may assume that $p = i{+}2$. Let $T = (\Delta')^{i-1}(Q(a))$ and $U = (\Delta')^{i-1}(Q(\alpha_i(a)))$. By Theorem 2.5 (a), we see $T = Q(a_i a_{i+1} a_{i+2})$ and $U = Q(\alpha(a_i a_{i+1} a_{i+2}))$. Since they are distinct, we see $T$ and $U$ have shape $(2,1)$. The promotion operator acts the same for entries $j < i$, so $Q(a) \mid_{\{i,i+1,i+2\}}$ and $Q(\alpha_i(a)) \mid_{\{i,i+1,i+2\}}$ are dual equivalent. Therefore $T = h_1(U)$. By inverting $(\Delta')^{i-1}$, we then see $Q(a)$ and $Q(\alpha_i(a))$ differ by the elementary dual equivalence $h_i$.

The proof of (b) is identical once we translate techniques into the appropriate setting. Here, Edelman-Greene insertion is replaced with Kraśkiewicz insertion and our notion of dual equivalence is shifted.

□

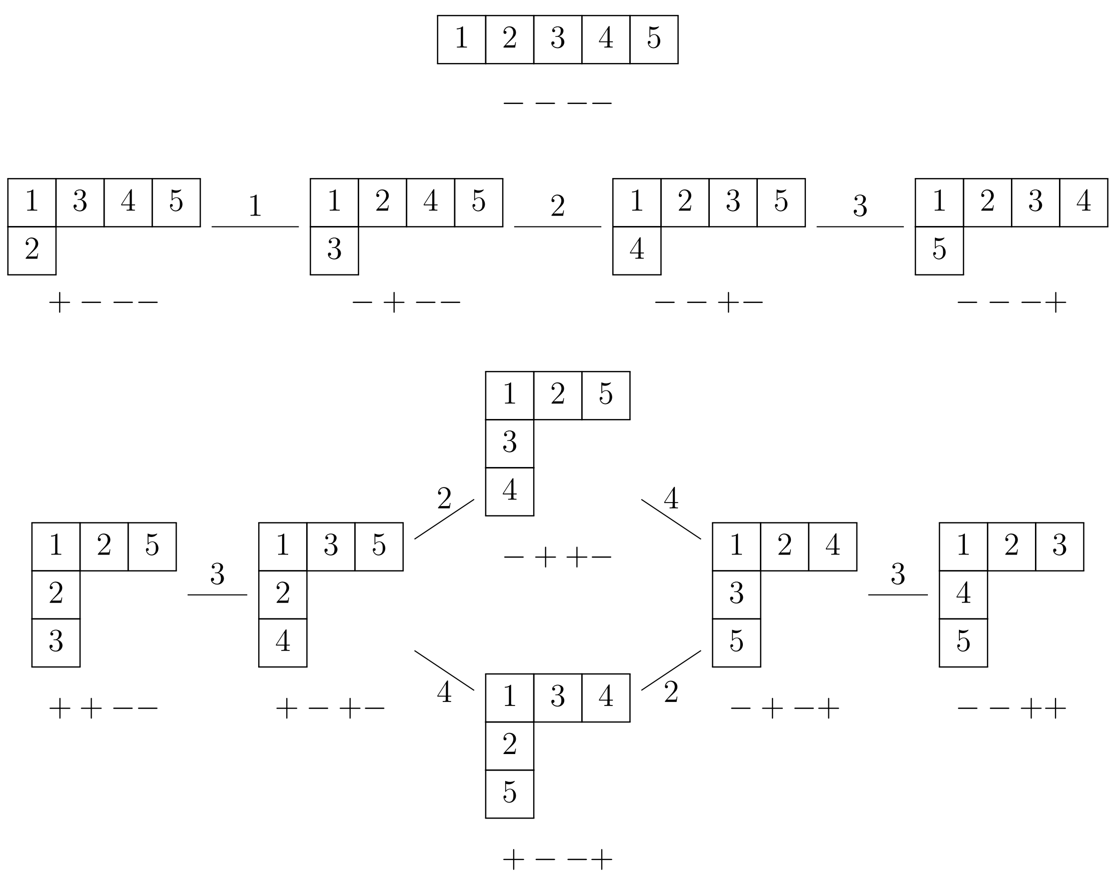
For Coxeter-Knuth moves of Type A, this result follows as a corollary of Reiner and Shimozono's plactification map, introduced in [33]. This map transforms a reduced word $a$ into a word $b$ in such a way that the Edelman-Greene insertion of $a$ and the RSK insertion of $b$ coincide. A nearly identical result is proved in [17] using a careful case-by-case analysis of individual Coxeter-Knuth moves under insertion. Similar analysis can be used to prove an analogous result for Type B. However, this approach is both much more tedious and much less illuminating.

## 2.4 Coxeter-Knuth and dual equivalence graphs

From Proposition 2.1 and Theorem 2.7, we see elementary dual equivalences span $SYT(\lambda)$ for every $\lambda$. In [3], Assaf studied the connected graph structure induced on $SYT(\lambda)$ by elementary dual equivalences and characterized such graphs by several properties. These properties were simplified by Roberts in [34]. The shifted analogue of this graph structure was developed independently in [2] and [6]. The approaches are slightly different, each with its own advantages. For $w \in B_\infty$ we will also define a similar graph structure on $R(w)$ called a Coxeter-Knuth graph.

We say a graph $\mathcal{G} = (V, E)$ is a *signed colored graph* of *degree* $n$ if there is a signature function $\sigma : V \to \{-,+\}^{n-1}$ and $E = E_1 \cup \cdots \cup E_{n-1}$.

**Definition 2.3.** For $\lambda$ a partition, the *elementary dual equivalence graph* $\mathcal{SG}_\lambda$ is the signed colored graph with vertex set $SYT(\lambda)$, signatures induced by the descent set and $U \sim_i T$ if $U = h_i T$ but $U \neq T$. Below are the elementary dual equivalence graphs of degree 5, up to conjugation (conjugation would invert the signatures).



$$\begin{array}{ccccccccc}
\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} & \overset{2}{\underset{3}{\text{———}}} & \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} & \underset{4}{\text{———}} & \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} & \underset{2}{\text{———}} & \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} & \overset{3}{\underset{4}{\text{———}}} & \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} \\
-+-- & & +-+- & & -++- & & -+-+ & & --+-
\end{array}$$

Here we omit $\sigma_5$, since it will always be $-$.

A *dual equivalence graph* is a finite disjoint union of elementary dual equivalence graphs of the same degree.

Morphisms on dual equivalence graphs must preserve edge labels and signatures. Given a signed colored graph $\mathcal{G} = (V, E)$ with signature $\sigma$ of degree $n$, the *restriction* of $\mathcal{G}$ to the interval $I = [i, j] \subset [n]$ is the graph on $V$ with edge set $E_i \cup \cdots \cup E_{j-1}$ and signature $\sigma\mid_I : V \to \{-,+\}^{j-i}$ where $(\sigma\mid_I)_T(k) = \sigma_T(i+k-1)$.

We associate functions to dual equivalence graphs using fundamental quasisymmetric functions. Given $\mathcal{G} = (V, E)$ a dual equivalence graph we define

$$F_{\mathcal{G}} = \sum_{T \in V} f_{\sigma_T}.$$

Let $\mathcal{G}_1, \ldots, \mathcal{G}_k$ be the connected components of $\mathcal{G}$ isomorphic to the elementary dual equivalence graphs of shapes $\lambda^1, \ldots, \lambda^k$ respectively. Then

$$F_{\mathcal{G}} = \sum_{i=1}^{k} \sum_{T \in V(G_i)} f_{\sigma_T} = \sum_{i=1}^{k} s_{\lambda^i} \tag{2.1}$$

so $F_{\mathcal{G}}$ is Schur positive, hence symmetric. Many symmetric functions have expansions in terms of fundamental quasisymmetric functions indexed by combinatorial objects. By demonstrating a dual equivalence graph structure on these combinatorial objects, a Schur positivity result is obtained. Moreover, identifying the components of this dual equivalence graph provides an explicit Schur basis decomposition. This has been done for many symmetric functions, such as Hall-Littlewood polynomials [35], certain Lascoux-Leclerq-Thibon polynomials [34], Kazhdan-Lusztig molecules of Type A [8] and much more.

In order to identify dual equivalence graphs, we investigate some of their properties.

**Proposition 2.3** ([3], Proposition 3.5)**.** *Let $\mathcal{G} = (V, E)$ be a dual equivalence graph of degree $n$ with signature $\sigma$ with $T, U, X, Y \in V$.*

*(a) For $T \in V$, if $\sigma_T(i{-}1) = -\sigma_T(i)$, then there exists $U \in V$ such that $T \sim_i U$.*

*(b) If $T \sim_i U$, then $\sigma_T(j) = -\sigma_U(j)$ for $j = i{-}1, i$ and $\sigma_T(j) = \sigma_U(j)$ for $j < i - 2$ and $j > i{+}1$.*

*(c) For $T \sim_i U$, if $\sigma_T(i-2) = -\sigma_U(i-2)$, then $\sigma_T(i-2) = -\sigma_T(i{-}1)$. Likewise, if $\sigma_T(i{+}1) = -\sigma_U(i{+}1)$, then $\sigma_T(i{+}1) = -\sigma_T(i)$.*

*(d) If $T \sim_i U \sim_j Y$ with $|i - j| > 2$, then there exists $X$ such that*

$$\begin{tikzcd}
 & U \arrow[dr, dash, "j"] & \\
T \arrow[ur, dash, "i"] \arrow[dr, dash, "j"'] & & Y \\
 & X \arrow[ur, dash, "i"'] &
\end{tikzcd} .$$

*(e) The restriction of $\mathcal{G}$ to any interval $[i, j] \subset [n]$ is a dual equivalence graph of degree $j - i{+}1$.*

*(f) For $T, U$ in same connected component of $\mathcal{G}$ and $i \in [n-1]$ there exists a path in $\mathcal{G}$ from $T$ to $U$ traversing an edge labeled $i$ at most once.*

Aside from the last, these follow from simple observations about dual equivalence graphs. These properties can be used to characterize dual equivalence graphs.

**Theorem 2.8** ([3], Theorem 3.9)**.** *Any signed colored graph satisfying the properties in Proposition 2.3 with property (e) for* $j - i \leq 3$ *is a dual equivalence graph.*

Note that property (f) is not local. In applications of this theorem, demonstrating this property proves to be the most difficult task.

**Theorem 2.9** ([34], Theorem 3.17)**.** *A signed colored graph satisfying properties (d) and (e) of Proposition 2.3, with (e) for* $j - i > 3$ *is a dual equivalence graph.*

Note property (e) implies properties (a), (b) and (c). This provides a totally local characterization of dual equivalence graphs. A similar result in a slightly less general case is proved by Chmutov in [8].

For shifted dual equivalence, the peak set serves the role that the descent plays in the unshifted case.

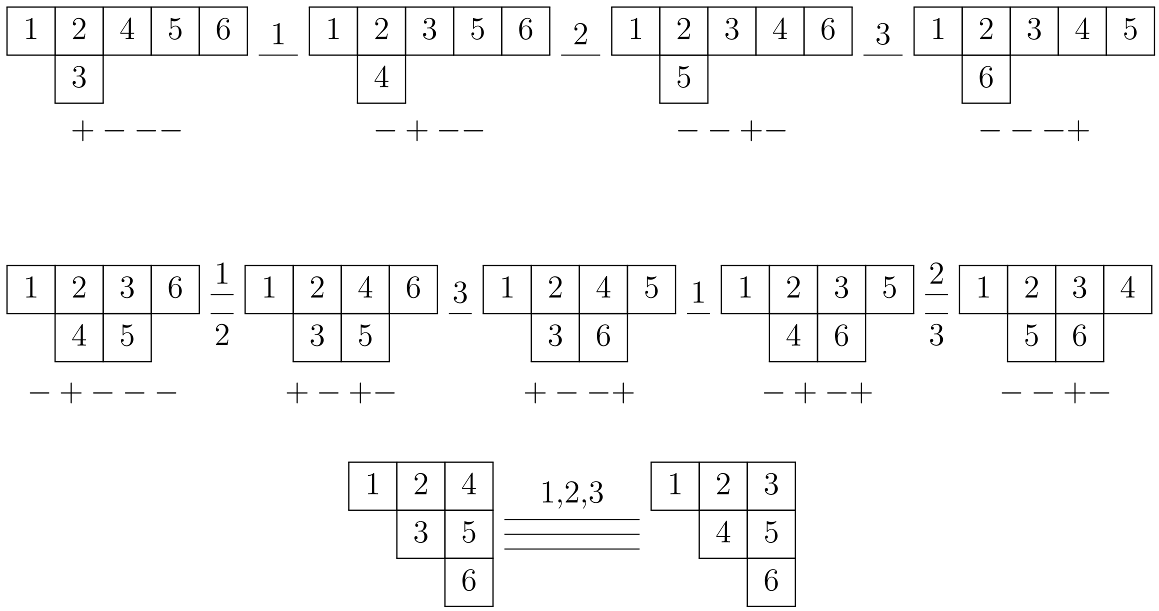
**Definition 2.4.** For $\lambda$ a strict partition, the *elementary shifted dual equivalence graph* of $\lambda$ is the signed colored graph with vertex set $ShSYT(\lambda)$, signatures induced by peak sets and $U \sim_i T$ if $U = h'_i T$ with $U \neq T$. Note the degree of this graph is one less than the size of $\lambda$.

A *shifted dual equivalence graph* is the finite disjoint union of elementary shifted dual equivalence graphs of the same degree.

Below are the shifted dual equivalence graphs of degree 6.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |

$$- - --$$




Here we omit $\sigma_5$, since it will always be $-$.

Again, we associate functions to shifted dual equivalence graphs using peak fundamental quasisymmetric functions. Given $\mathcal{G} = (V, E)$ a shifted dual equivalence graph we define

$$G_{\mathcal{G}} = \sum_{T \in V} \Theta_{\sigma'_T}.$$

Let $\mathcal{G}_1, \ldots, \mathcal{G}_k$ be the connected components of $\mathcal{G}$ isomorphic to the elementary shifted dual equivalence graphs of shapes $\lambda^1, \ldots, \lambda^k$ respectively. Then

$$G_{\mathcal{G}} = \sum_{i=1}^{k} \sum_{T \in V(G_i)} \Theta_{\sigma'_T} = \sum_{i=1}^{k} Q_{\lambda^i}$$

so $G_{\mathcal{G}}$ is $Q$-Schur positive, hence symmetric. Therefore by identifying a shifted dual equivalence graph structure on a collection of combinatorial objects, we may associate a projective symmetric function to them.

Shifted dual equivalence graphs have properties similar to those of unshifted dual equivalence graphs. We defer a more detailed treatment to Chapter 5.

The graph structure induced on reduced words by Coxeter-Knuth moves is quite similar.

**Definition 2.5.** For $w \in S_\infty$, the *Type A Coxeter-Knuth graph* on $w$ has vertex set $R(w)$, signatures induced by descent sets and $a \sim_i b$ if $b = \alpha_i a$ but $a \neq b$.

For $w \in B_\infty$, the *Type B Coxeter-Knuth graph* on $w$ has vertex set $R(w)$, signatures induced by peak sets and $a \sim_i b$ if $b = \beta_i a$ but $a \neq b$.

From Proposition 2.2, we know that Coxeter-Knuth moves act on recording tableaux as elementary dual equivalences.

**Theorem 2.10.** *Coxeter-Knuth graphs are (shifted) dual equivalence graphs.*

*Proof.* Let $\mathcal{G}_w = (V, E)$ be a Coxeter-Knuth graph associated with $w$ of Type A or Type B. From Theorems 2.2 and 2.4, we see the connected components of $\mathcal{G}$ are words in $R(w)$ with same recording tableaux. By Theorems 2.1 and 2.3 we can associate to each word its recording tableau without effecting the signature. Therefore Proposition 2.2 implies that each connected component is a (shifted) dual equivalence graph.

□

This result was first observed by Billey for Type A (see [17]) and in [6] for Type B. However, the first approach in Type A was less direct, relying on recurrences derived from the Little map. As a corollary, we recover the fact that the Type A Stanley symmetric functions are Schur positive and the Type B Stanley symmetric functions are $Q$-Schur positive.

In Chapter 5, we demonstrate an analogue of Theorem 2.9 for shifted dual equivalence graphs following the treatment in [6]. In [2], Assaf has obtained an analogue of Theorem 2.8.

## 2.5 Inverse maps for the longest element

In general, insertion algorithms can be inverted by identifying the last box added and inverting the insertion row by row. However, for reduced words of the reverse permutation $w_0 = n(n-1)\dots 1$ Edelman and Greene established a particularly beautiful inverse map. Note $\ell(w_0) = N := \binom{n}{2}$. Let $a = a_1 \dots a_N \in R(w_0)$. Since $P(a)$ is row and column strict with $N$ entries, the largest of which is $n-1$, we see

$$P(a) = \begin{array}{|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & \cdots & n-2 & n-1 \\
\hline
2 & 3 & 4 & \cdots & n-1 \\
\cline{1-5}
\vdots & \vdots & \vdots \\
\cline{1-2}
n-2 & n-1 \\
\cline{1-2}
n-1 \\
\cline{1-1}
\end{array} .$$

Therefore $Q : R(w_0) \to SYT(\lambda_0)$ is a bijection where $\lambda_0$ is the *staircase* shape $(n-1, n-2, \dots, 1)$.

We now characterize the inverse of this map in terms of $\Delta^{-1}$. For $T$ a standard Young tableaux, let $c(T)$ be the column-index of the largest label in $T$. Let $T \in SYT(\lambda_0)$. We define the *inverse Edelman-Greene map* $\Gamma$ by

$$\Gamma(T) = c(\Delta^{-(N-1)}(T)\ c(\Delta^{-(N-2)}(T))\ \dots\ c(\Delta^{-1})(T))\ c(T).$$

**Theorem 2.11** ([9], Theorem 7.18)**.** *Let $T \in SYT(\lambda_0)$. Then $\Gamma(T) \in R(w_0)$ and $Q$*

*and* $\Gamma$ *are inverses.*

*Example* 2.4. Let

$$T \;=\; \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}\;.$$

We compute $\Gamma(T)$. First, we list $T, \Delta^{-1}(T), \Delta^{-2}(T), \Delta^{-3}(T), \Delta^{-4}(T)$ and $\Delta^{-5}(T)$.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}\;.$$

Then $\Gamma(T) = 132312 \in R(4321)$.

The longest element in $B_{n-1}$ is $w_0 = \bar{1}\bar{2}\ldots\bar{n}$. Now $\ell(w_0) = N = n^2$. Let $a \in R(w_0)$. Since $P(a)$ is row and column strict with $N$ entries, the largest of which is $n-1$, we see

$$P(a) = \begin{array}{ccccccccc} \boxed{n-1} & \boxed{n-2} & \boxed{n-3} & \cdots & \boxed{0} & \cdots & \boxed{n-3} & \boxed{2n-2} & \boxed{n-1} \\ & \boxed{n-3} & \boxed{n-4} & \cdots & \boxed{1} & \cdots & \boxed{n-2} & \boxed{n-1} & \\ & & \ddots & & \vdots & & \iddots & & \\ & & & & \boxed{n-1} & & & & \end{array}\;.$$

Therefore $Q' : R(w_0) \to SYT(\lambda_0')$ is a bijection where $\lambda_0'$ is the *shifted staircase* shape $(2n-1, 2n-3, \ldots, 1)$. The inverse takes a nearly identical form. The only difference is that we must subtract $n$ from the column-index so that we take values in $0, \ldots, n-1$. We denote this by $c'(T)$. For $T \in SYT(\lambda_0')$, the *inverse Kraśkiewicz map* $\Gamma'$ is defined by

$$\Gamma'(T) = c'(\Delta^{-(N-1)}(T))\ c'(\Delta^{-(N-2)}(T))\ \ldots\ c'(\Delta^{-1}(T))\ c(T).$$

**Theorem 2.12** ([16, 22]). *Let $T \in SYT(\lambda_0')$. Then $\Gamma'(T) \in R(w_0)$ and $Q'$ and $\Gamma$ are inverses.*

Proctor conjectured that $\Gamma'$ is a bijection. This was first proved in by Haiman in [16, Theorem 5.12], before Kraśkiewicz insertion was defined. That the maps coincide is a result of Kraśkiewicz.

We now present many other bijections from $R(w_0)$ to $SYT(\lambda_0)$. To do so, we must first introduce an alternative definition of promotion. For $T \in SYT(\lambda)$ with $\lambda \vdash n$, the *toggle operator* $t_i$ acts on $T$ by exchanging the labels $i$ and $i{+}1$ so long as the resulting tableau is standard. Note $t_i^2 = 1$. Then $\Delta = t_{n-1}t_{n-2}\ldots t_1$ and $\Delta^{-1} = t_1t_2\ldots t_n$.

*Example* 2.5. We apply the sequence of toggles $t_1 \ldots t_5$ to the following tableau.

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6\\\cline{1-2} 5\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}\,.$$

Note $t_5$ and $t_4$ act non-trivially, but $t_3$ acts trivially. This has the effect of replacing 6 with 4, which is equivalent to sliding 3 into the position of 6, then incrementing. The end result is $\Delta^{-1}(T)$, as computed in the continuation of Example 2.3.

## 2.5 Inverse maps for the longest element

Let the group $G$ act on the set $S$. Each group element $g$ induces an orbit structure $\langle g \rangle$ on $X$. For $g_1, g_2 \in G$ such that $g_2 = g g_1 g^{-1}$, the map $x \mapsto gx$ is an equivariant bijection between $\langle g_1 \rangle$ and $\langle g_2 \rangle$. This can be seen from the following diagram,

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "g_2"'] & X \arrow[d, "g_1"] \\ X \arrow[r, "g"'] & X \end{tikzcd}.$$

For $w \in S_{n-1}$, we define $\Delta_w = t_{w_1} t_{w_2} \ldots t_{w_{n-1}}$. Based on the above observation, Striker and Williams showed in [40] that when $\Delta$ and $\Delta_p i$ are conjugate they induce the same orbit structure on $SYT(\lambda_0)$. Note the toggle operators $t_i$ generate a group $G$.

**Lemma 2.1.** *Let $G$ be a group with generators $g_1, \ldots, g_n$ such that $g_i^2 = 1$ and $g_i g_j = g_j g_i$ for $|i - j| > 1$. Then for $w, v \in S_n$ the group elements $\prod_{i=1}^n g_{w_i}$ and $\prod_{j=1}^n g_{v_j}$ are conjugate.*

*Proof.* It suffices to prove that $\prod_{i=1}^n g_i$ is conjugate to $\prod_{j=1}^n g_{w_j}$ for all $w$. Let $k$ be the largest index in $\prod_{j=1}^n g_{w_j}$ that can be pushed all the way to the left using the commutativity relation. Then conjugate by $k$, and push the new $k$ on the right as far to the left as possible. It will be stopped by either $k+1$ or $k-1$. If it has been stopped by $k-1$, the number of inversions in $w$ has been decreased by at least 1. If it has been stopped by $k+1$, the inversion with values $k$ and $k-1$ has been replaced by the inversion with values $k$ and $k+1$. Then each step of this process either decreases the number of inversions or increases the value of an inversion. Therefore, it must eventually terminate at $\prod_i^n g_i$.

□

This result appears in [19], while our proof follows the proof presented in [40]. Applying it to the toggle group, we see $\Delta$ and $\Delta_w$ are conjugate for all $w \in S_{n-1}$. Therefore, $\Delta_w$ induces the same orbit structure on $SYT(\lambda_0)$ as $\Delta$ does. Moreover, we can conjugate from one to the other such that each intermediary step is of the form $\Delta_v$ for some $v \in S_{n-1}$.

Define

$$\Gamma_w(T) = c(\Delta_w^{-(N-1)}(T)\ c(\Delta_w^{-(N-2)}(T))\ \dots\ c(\Delta^{-1})_w(T))\ c(T).$$

Nathan Williams conjectured (see [41, p48]) that for the permutation $w = 135\dots\ 246\dots$ the map $\Gamma_w$ provides a bijection between $R(w_0)$ and $SYT(\lambda_0)$. In fact, this holds for any $w \in S_n$.

**Theorem 2.13.** *Let $T \in SYT(\lambda_0)$ and $w \in S_{N-1}$. Then $\Gamma_w$ is injective and $\Gamma_w(T) \in R(w_0)$.*

*Proof.* Note $\Gamma = \Gamma_{w_0}$ is a bijection from $SYT(\lambda_0)$ to $R(w_0)$. By Lemma 2.1, we need only show that if $\Gamma_v$ is a bijection then $\Gamma_{t_ivt_i}$ is a bijection as well. We begin by observing that

$$\begin{aligned}\Gamma_{t_ivt_i} &= c((t_i\Delta_vt_i)^{-(N-1)}T)\ c((t_i\Delta_vt_i)^{-(N-2)}T)\ \dots\ c((t_i\Delta_vt_i)^{-1}T)\ c(T)\\ &= c(t_i\Delta_v^{-(N-1)}t_iT)\ c(t_i\Delta_v^{-(N-2)}t_iT)\ \dots\ c(t_i\Delta_v^{-1}t_iT)\ c(T)\end{aligned}$$

as $t_i^2 = 1$.

For $i < N-1$, note the $c(t_iT) = c(T)$ for all $T$. Then

$$\Gamma_{t_ivt_i} = c(t_i\Delta_v^{-(N-1)}t_iT)\ c(t_i\Delta_v^{-(N-2)}t_iT)\ \dots\ c(t_i\Delta_v^{-1}t_iT)\ c(T)$$

$$= c(\Delta_v^{-(N-1)} t_i T)\ c(\Delta_v^{-(N-2)} t_i T)\ \ldots\ c(\Delta_v^{-1} t_i T)\ c(T) \qquad = \Gamma_v(t_i T),$$

so we see $\Gamma_{t_i v t_i}$ is a bijection as well.

For $i = N-1$, the value of $c(t_i T)$ determines the location of $N-1$ in $T$. Then

$$\begin{aligned}\Gamma_{t_i v t_i} &= c(t_i \Delta_v^{-(N-1)} t_i T)\ c(t_i \Delta_v^{-(N-2)} t_i T)\ \ldots\ c(t_i \Delta_v^{-1} t_i T)\ c(T) \\ &= c(t_i \Delta_v^{-(N-1)} (t_i T)\ c(t_i \Delta_v^{-(N-2)} (t_i T)\ \ldots\ c(t_i \Delta^{-1})_v (t_i T)\ c(t_i(t_i T)).\end{aligned}$$

The behavior of $\Gamma_{t_i v t_i}$ depends on the order of $N-2$ and $N-1$ in $v$. If $N-2$ appears before $N-1$ in $v$, then $c(t_{N-1} T) = c(\Delta_v(T))$. If $N-2$ appears after $N-1$ in $v$, then $c(t_{N-1} T) = c(\Delta_v^{-1}(T)$. Therefore

$$\Gamma_{t_i v t_i}(T) = \Gamma_v(\Delta_v t_{N-1} T) \ \text{ or } \ \Gamma_{t_i v t_i}(T) = \Gamma_v(\Delta_v^{-1} t_{N-1} T).$$

In either case, we see $\Gamma_{t_i v t_i}(T)$ is a bijection. This completes our proof.

$\square$

All of the same assumptions hold for signed permutations. Therefore, we also obtain many variants of $\Gamma'$.

# Chapter 3

# Little maps

In this Chapter, we define Little maps for Type A and Type B finite Coxeter groups. These maps act on a reduced word by a sequence of modifications called *Little bumps* until is is reduced words of (isotropic) Grassmannian permutations. There is a simple bijection between reduced words of Grassmannian permutations and standard Young tableaux, ordinary for Type A and shifted for Type B. The Type A Little map was introduced in [27], while the Type B Little map was introduced in joint work with Billey, Roberts and Young [6].

## 3.1 Little bumps

We begin by introducing Little bumps. These modify a reduced word of one permutation to a reduced word of another. While defined on reduced words, Little bumps are best understood as acting on wiring diagrams. For $a = a_1 \dots a_p \in R(w)$ where $w \in B_\infty$, recall $w^k(a) = s_{a_1} \dots s_{a_k}$ so $w^p = w$ and $w^0$ is the identity. Recall the *wiring diagram* of $a$ is the diagram on $(\{0, 1, \dots, p\} \times \mathbb{Z} \setminus \{0\}$ where $(w_j, w_i)$ is labeled with $(w^i)^{-1}(j)$ and entries with the same label are connected from left to right. We suppress entries with $|j| > \max\{a_i\}$, and for Type A we suppress entries with $j < 0$ as well. Similarly, recall the *trajectory* of $i$ in $a$ is the sequence $\{(w^j)^{-1}(i)\}_{j=0}^{p}$.

We now set out to define Little bumps, but first require some notation. For $a = a_1 \dots a_p \in R(w)$ where $w \in B_\infty$ and $\delta \in \{-1, 1\}$, we define *push* $P_i^\delta$ at index $i$ to be the map that adds $\delta$ to $a_i$ while fixing the rest of the word. We will write $P_k^-$ and $P_k^+$ for $\delta = -1, 1$ respectively. The effect pushes have on wiring diagrams can be observed in Example 3.1. If $a = a_1 \dots a_p$ is a word that is not reduced, we say a *defect* is caused by $a_i$ and $a_j$ with $i \neq j$ if the removal of either leaves a reduced word. The following lemma can be deduced for Types A and B from the wiring diagrams, but it holds more generally for Coxeter groups.

**Lemma 3.1** (Lemma 21, [25])**.** *Let $a = a_1 \dots \hat{a}_i \dots a_p \in R(w)$ such that $a_1 \dots a_p$ is not reduced. Then there exists a unique $j \neq i$ such that $a_1 \dots \hat{a}_j \dots a_p$ is reduced. Moreover, $a_1 \dots \hat{a}_j \dots a_p \in R(w)$.*

Let $a = a_1 \dots a_p$ be a reduced word of $w \in B_\infty$ and $(w_j, w_i) \in \mathrm{Inv}_B(w)$ such that $l(wt_{ij}) = l(w) - 1$. Recall $w^k(a) = s_{a_1} s_{a_2} \dots s_{a_k}$. Fix $\delta \in \{-1, 1\}$. We define the *Little bump for $w$* at $w_j, w_i)$ in the direction $\delta$, denoted $B^\delta_{(w_j, w_i)}$, as follows

(a) Identify $k$ such that $a_k$ introduces the inversion $(w_j, w_i)$. If $(w^k)^-1(a_k) = w_i$, set $b = P_k^\delta(a)$. Otherwise $w^k(a_k) = -w_i$, and we set $b = P_k^{-\delta}(a)$.

(b) If $b$ is reduced, return $b$. Otherwise, by Lemma 3.1 there is a unique defect caused by $b_k$ and some $b_l$ with $l \neq k$.

- If $(w^k)^{-1}(b_k) = (w^{l+1})^{-1}(b_l)$, set $b := P_l^\delta(b)$, $k = l$ and repeat Step (2).

- If $(w^k)^{-1}(b_k) = -(w^{-1})^{l+1}(b_l)$, set $\delta = -\delta$, $b := P_l^\delta(b)$, $k = l$ and repeat Step (2).

**Remark 3.1.** At every step, the pushes move the crossings consistently in the initial direction of $\delta$. In the first step, we move the $(w_j, w_i)$-crossing in the wiring diagram up if $\delta = +1$ and down if $\delta = -1$. If wires $w_j$ and $w_i$ cross in the upper half plane then the swap $a_k$ is replaced with $a_k + \delta$. However, if wires $w_j, w_i$ cross in the lower half plane then the swap $a_k$ is replaced with $a_k - \delta$, so the sign of $\delta$ is switched. If a new defect crossing is later found on the other side of the $x$-axis from the last crossing, then the sign of $\delta$ will switch again so that the crossing continues to move in the same direction. Thus, if the initial push moved $(w_j, w_i)$ down, each subsequent iteration will continue to move a crossing down, but the effect on the word from the corresponding push can vary.

**Remark 3.2.** Observe that in each iteration of Step 2, the word $b$ has the property that its subword $b_1 b_2 \dots \widehat{b_k} \dots b_p$ is reduced.

When $a = a_1 \dots a_p$ is a Type A reduced word, the definition of a Little bump needs one modification. For $b = B^\delta_{(w_j,w_i)}(a)$, if $b_j = 0$ for some $j$ then instead return $B^\delta_{(i+1,j+1)}((a_1+1)\dots(a_p+1))$. This ensures the resulting word remains a Type A reduced word. Therefore, we can prove properties of Little bumps applied to Type A and Type B reduced words simultaneously.

We present an algorithm equivalent to Little's original algorithm in [27] based on our Type B algorithm. To do so, we remove cases that cannot occur for Type A reduced words. Let $a = a_1 \dots a_p$ be a reduced word of $w \in S_\infty$ and $(w_j, w_i) \in \mathrm{Inv}_B(w)$ such that $l(wt_{ij}) = l(w) - 1$.

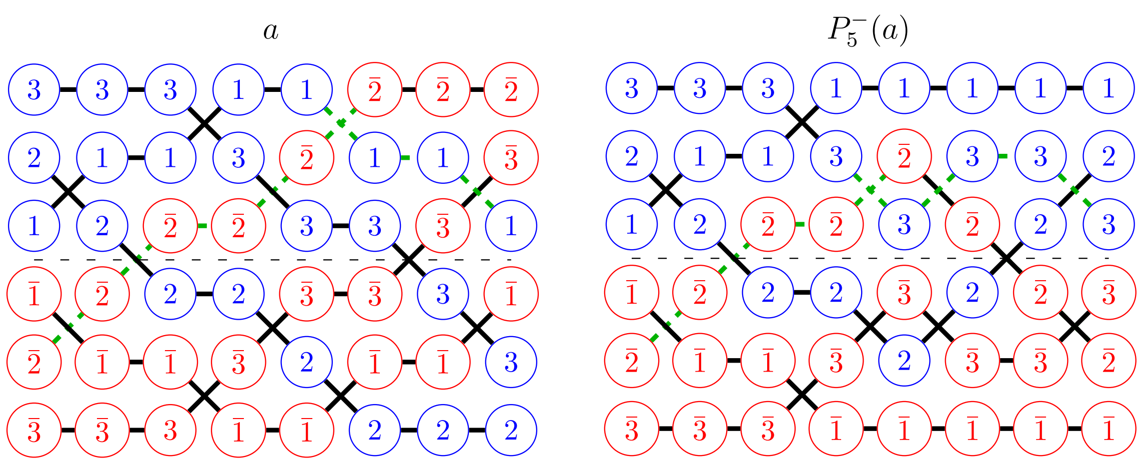
(a) Identify $k$ such that $a_k$ introduces the inversion $(w_j, w_i)$, and set $b = P_k^{\delta}(a)$.

(b) If some $b_i = 0$, set $b := (b_1 + 1) \ldots (b_p + 1)$.

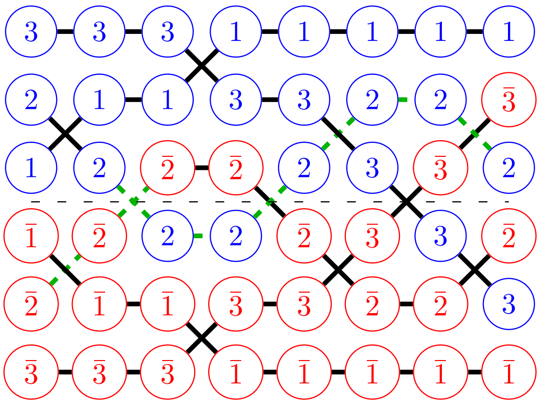
(c) If $b$ is reduced, return $b$. Otherwise, by Lemma 3.1 there is a unique defect caused by $b_k$ and some $b_l$ with $l \neq k$. Set $b := P^{\delta_l(b)}$ and return to (2).

From the definition, it is not immediately clear that Little bumps terminate. To show this, we will need to track where the next push can occur. The *boundary* of the $(w_j, w_i)$-crossing in column $k$ of the word $a$, denoted $\partial^k_{(w_j,w_i)}(a)$, is the union of the trajectory of $i$ from columns 0 to $k$ and the trajectory of $j$ from columns $k$ to $n$. Note any defect resulting from moving the $(w_j, w_i)$-crossing down will involve the boundary. In Example 3.1, the boundary of of each crossing that will be pushed is dashed.

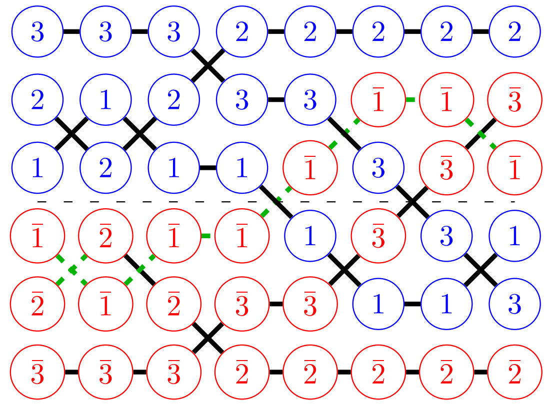
*Example* 3.1. The sequence of pushes corresponding to the Little bump $B^-_{(-2,1)}$ as applied to $a = 1021201 \in R([\bar{1}, 3, 2])$. The boundary of each crossing about to be moved is dashed. Here, red is negative and blue is positive.



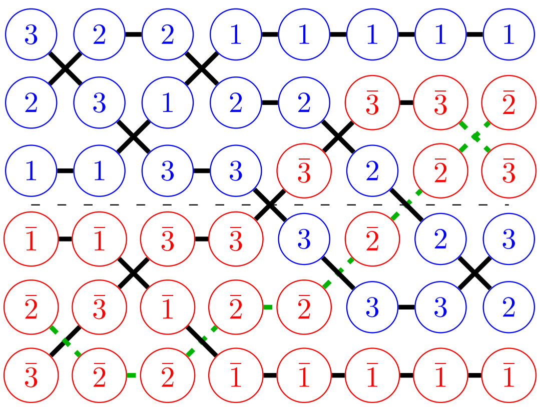
$P_4^- P_5^- (a)$



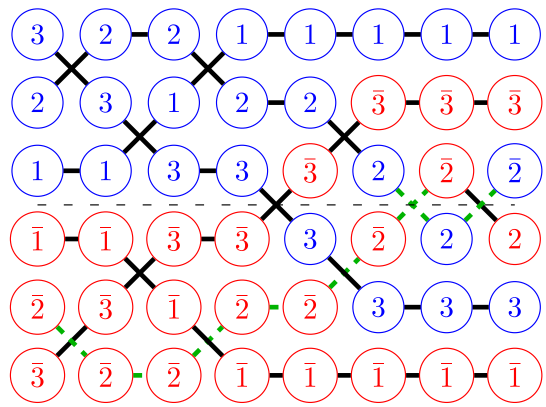

$P_2^+ P_4^- P_5^- (a)$



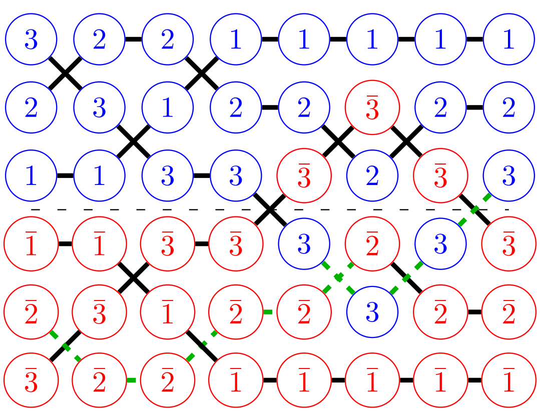

$P_1^+ P_2^+ P_4^- P_5^- (a)$



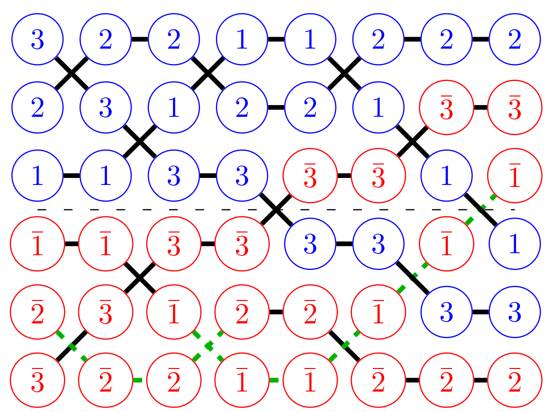

$P_7^- P_1^+ P_2^+ P_4^- P_5^- (a)$



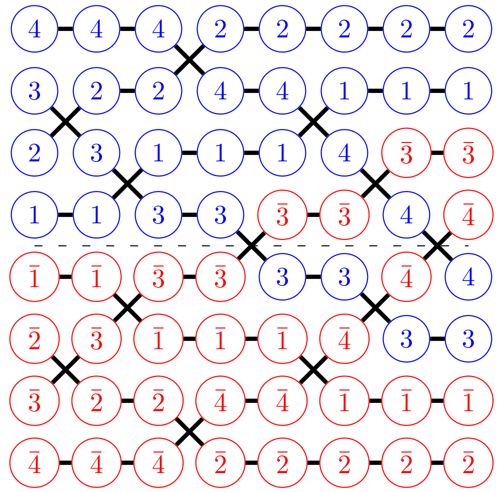

$P_6^+ P_7^- P_1^+ P_2^+ P_4^- P_5^- (a)$




$P_5^+ P_6^+ P_7^- P_1^+ P_2^+ P_4^- P_5^- (a)$



$$B^-_{(-2,1)}(a) = P_3^+ P_5^+ P_6^+ P_7^- P_1^+ P_2^+ P_4^- P_5^-(a)$$



**Lemma 3.2.** *Let $a \in R(w)$ and $B^\delta_{(w_j,w_i)}$ be a Little bump for $w$ consisting of the sequence of pushes $P^{\delta_1}_{t_1}, P^{\delta_2}_{t_2}, \ldots$ acting on $a$. Then for all $k$ and $\delta \in \{-1, 1\}$ the push $P^\delta_k$ appears at most once in this sequence. Hence, the Little bump algorithm terminates in a finite number of pushes.*

*Proof.* This proof is essentially the same as the proof in Lemma 5 in [27]. Since $B^-_{(w_j,w_i)} = B^+_{(-j,-i)}$, we need only demonstrate the result when $\delta = -1$.

Let $a = a_1a_2\ldots a_p \in R(w)$ and $a_k$ denote the swap introducing the inversion $(w_j, w_i)$. We will track the Little bump $B^-_{(w_j,w_i)}$, choosing $i < j$ so that the bump starts with the push $P^\delta_k$ that moves the $(w_j, w_i)$-crossing down. Any defect introduced by this push must occur along the boundary $\partial^k_{(w_j,w_i)}(a)$. Either $b = P^\delta_k(a)$ is reduced or there is some $l \neq k$ such that $b_k$ and $b_l$ cause a defect. Suppose the latter case holds. Then $b_k$ and $b_l$ swap the wire $i$ with some wire $h \neq j$. By considering the reverse of the word if necessary, we may assume $l < k$. Observe that $\partial^k_{(w_j,w_i)}(a)$ and $\partial^l_{(i,h)}(b)$ coincide from 1 to $l$ and from $k$ to $n$. Moreover, between $l$ and $k$, the boundary $\partial^l_{(i,h)}(b)$ is strictly lower in the wiring diagram than $\partial^k_{(w_j,w_i)}(a)$. This can be seen in each step of the Little bump shown in Example 3.1.

Observe that the trajectories of $-w_j$ and $-w_i$ will not interact with $\partial^l_{(i,h)}(b)$ unless $w_i = -w_j$. Therefore the boundary is monotonically decreasing. Next, we verify that no push occurs twice. In particular, we claim a crossing in some column $k$ can never participate in a defect twice.

We examine a single push $P_k^\delta$, appearing for the first time in a Little bump $B$, that moves the $(w_j, w_i)$-crossing in the word $a$. Let $b = P_k^\delta(a)$ so $b_k$ introduces some crossing $(w_j, h)$ in the wiring diagram of $b$. Without loss of generality assume for any resulting defect between $b_k$ and $b_l$ that $l$ is to the right of $k$. Note if no such defect exists then the resulting word is reduced and no additional pushes occur. Observe $\partial^l_{(w_j,h)}(b)$ is strictly below the trajectory of $i$ in columns between $k$ and $l$ and strictly below the trajectory of $h$ in columns to the right of $l$. This can be seen in Example 3.1, for example in the second and third wire diagrams depicted. Therefore no future defect involving the initial crossing in column $k$ can be in a column to the right of $k$. Any future defect featuring the initial crossing in column $k$ must feature a crossing in a column to the left of $k$. Until this occurs, one of the trajectories in the initial crossing is strictly below the sequence of boundaries to the left of $k$. Eventually, some crossing to the left of column $k$ must be moved down. When this occurs, the resulting boundary will move strictly below the initial crossing. For the crossing in the second wire diagram of Example 3.1, this occurs between the fourth and fifth wire diagrams. Therefore, the $(w_j, w_i)$-crossing in column $k$ can never be moved in the same direction twice in the same Little bump. This implies no push occurs more than once. $\square$

For Type B reduced words, $P_k^-$ and $P_k^+$ can appear in the same Little bump. For example, $P_5^-$ and $P_5^+$ both occur in Example 3.1.

**Corollary 3.1.** *Let $w \in B_\infty$ with $a = a_1 \dots a_p \in R(w)$ and $B^\delta_{(w_j,w_i)}$ be a Little bump for $w$. Then*

$$peaks(a) = peaks(B^\delta_{(w_j,w_i)}(a)).$$

*Proof.* Say $k \in peaks(a)$, so $a_{k-1} < a_k > a_{k+1}$. A push applied to one of these entries may leave an equality in the resulting word $b$, say $b_{k-1} = b_k$. In this case, there is a defect caused by $b_{k-1}$ and $b_k$, so we push the other next. The direction of the new push for defects caused by adjacent entries will be the same unless $b_{k-1} = b_k = 0$. This cannot occur since $a_k > a_{k-1} \geq 0$ and if $a_k = 1$, then $a_{k-1} = 0 = a_{k+1}$ so $a_1 \dots \hat{a_k} \dots a_p$ is not reduced hence $P_k^-$ could not be applied by the hypothesis for the Little bump algorithm. □

For Type A reduced words, a more refined statement is possible, as observed in [27].

**Corollary 3.2.** *Let $w \in A_\infty$ with $a = a_1 \dots a_p \in R(w)$ and $B^\delta_{(w_j,w_i)}$ be a Little bump for $w$. Then*

$$des(a) = des(B^\delta_{(w_j,w_i)}(a)).$$

*Proof.* From the simplified algorithm, we see the direction of each push is $\delta$. Let $k \in des(a)$ so that $a_k > a_{k+1}$ and set $b = B^\delta_{(w_j,w_i)}(a)$. By Lemma 3.2, we see each $a_j$ is modified at most once so $b_i \geq b_{i+1}$. Since $b_k \neq b_{k+1}$ we see $b_k > b_{k+1}$. Similarly, for $k \notin des(a)$, we see $b_ik < b_{k+1}$.

□

We now set out to characterize the domain and range of Little bumps. For $v \in$

$A_\infty, B_\infty$ and $j \in \mathbb{Z} - \{0\}$, we define

$$D(v,j) = \{vt_{ij} : \; i < j \text{ and } l(vt_{ij}) = \ell(v) + 1\}$$
$$U(v,j) = \{vt_{jk} : \; j < k \text{ and } l(vt_{jk}) = \ell(v) + 1\}.$$

Note $D(v,-j) = U(v,j)$. For $v \in A_\infty$, if $D(v,j)$ would be empty, it instead becomes $D(1 \oplus v, j+1)$. We now prove [27, Theorem 3] and its analogue for the Type B Little map, from which we can deduce Theorems 1.4 and 1.6.

**Theorem 3.1.** *Let $v \in B_\infty$ and $j \neq 0$. Then*

$$\sum_{y \in D(v,j)} |R(y)| = \sum_{x \in U(v,j)} |R(x)|.$$

*Proof.* We will prove the equality bijectively by using a collection of Little bumps. Define a map $M_{w,j}$ on $\cup_{x \in U(v,j)} R(x)$ as follows. Say $a = a_1 \dots a_p \in R(x)$ for some $x \in U(v,j)$. Then $x = vt_{jk}$ for some unique $k > j$. Furthermore, $B^-_{(w_k,w_j)}$ is a Little bump for $x$.

When applying $B^-_{(w_k,w_j)}$ on $a$, the initial push is some $P^{\delta_1}_{l_1}$ with $a_1 \dots \widehat{a_{l_1}} \dots a_p \in R(v)$. Let $b = P^{\delta_1}_{l_1}(a)$. Then, $s_{b_1} \cdots s_{b_p} = vt_{ji_1}$ for some $i_1 \neq j, k$.

If $b$ is reduced, the bump is done. Otherwise, by Lemma 3.1, there is some unique defect between $b_{l_1}$ and $b_{l_2}$ so we push in column $l_2$ next. Then $b_1 \dots \widehat{b_{l_2}} \dots b_p \in R(v)$. Thus, recursively $B^-_{(w_k,w_j)}(a) \in R(vt_{ji})$ for some $i \neq j, k$.

Assume for the sake of contradiction that $i > j$, and say the $(w_j, w_i)$-crossing in the wiring diagram of $c = B^-_{(w_k,w_j)}(a)$ occurs in column $l$. By removing the $l$th swap from $c$ we get a wiring diagram for $v$ that does not have $(w_i, w_j)$ as an inversion. Thus, the $w_i$-wire must stay entirely above the $w_j$-wire. Here the $w_i$ wire is above the boundary of the last push, so it cannot be a part of the final push. We can then conclude that $c \in R(vt_{ij})$ for some $i < j$. Since $l(vt_{ij}) = l(vt_{jk})$, we see $vt_{ij} \in D(v,j)$. Thus,

$$M_{w,j} : \cup_{x \in U(v,j)} R(x) \longrightarrow \cup_{y \in D(y,j)} R(y).$$

Since the Little bump algorithm is reversible and $B^+_{(w_j,w_i)}(c) = a$ in the notation above, we know $M_{w,j}$ is injective.

The bijective proof is completed by observing that $D(v,j) = U(v,-j)$, $U(v,j) = D(v,-j)$ and $B^-_{(-j,-i)} = B^+_{(w_j,w_i)}$ is a Little bump for $vt_{ij} \in D(v,j)$ whose image, by the above argument, is a reduced word of some $x \in U(v,j)$. □

Note the same result holds for both Type A and Type B Little bumps.

Recall the notation of transition equations from Section 1.4. If $w$ is not increasing, let $(r < s)$ be the lexicographically largest pair of positive integers such that $w_r > w_s$. Set $v = wt_{rs}$. Let $T(w)$ be the set of all signed permutations $w' = vt_{ir}$ for $i < r$ such that $\ell(w') = \ell(w)$. We now show that the *canonical* Little bump $B^-_{(w_s,w_r)}$ for $w$ respects the transition equations in Theorems 1.4 and 1.6.

**Corollary 3.3.** *Let $a \in R(w)$ and $B^-_{(r,s)}$ be the* canonical Little bump *for $w$. Recall that $(r,s)$ is the lexicographically last inversion in $w^{-1}$. Then $B^-_{(w_s,w_r)}(a)$ is a reduced word for $w'$ where*

$$w' \in T(w) = \{wt_{rs}t_{lr} \mid l < r \text{ and } l(w) = l(wt_{rs}t_{rl})\}.$$

*Proof.* Observe $U(wt_{rs}, r) = \{w\}$ and

$$D(wt_{rs}, r) = \{wt_{rs}t_{lr} \mid l < r \text{ and } l(w) = l(wt_{rs}t_{rl})\} = T(w).$$

The result follows from Lemma 3.1. □

*Example* 3.2. The permutation 24153 has canonical bump $B^{-}_{(3,5)}$. We have

$$R(24153) = \{3412, 3142, 1342, 1324, 3124\},$$

whose image under $B^{-}_{(3,5)}$ is

$$\{2313, 2132, 1232, 1323, 3123\} = R(34125) \cup R(24315).$$

## 3.2 Little maps

The Type A and Type B Little maps follow the transition equations of [26] and [4] respectively. These transition equations, proved in the setting of Schubert polynomials, are recurrences on Stanley symmetric functions. Through repeated application the transition equations terminate in Grassmannian elements, whose Stanley symmetric functions are Schur and P-Schur functions, respectively.

Recall a permutation is *Grassmannian* if it has only one descent. In the case of a signed permutation, this means the image of $[n]$ is an increasing sequence. Such signed permutations are *isotropic Grassmannians*. We first describe a simple bijection called Tab between reduced words of Grassmannian permutations and standard Young tableaux. Next, we show how to extend Tab to a map $\mathrm{Tab}_B$ between reduced words of isotropic Grassmannian permutations and shifted standard Young tableaux.

Let $a = a_1 \ldots a + p \in R(w)$ where $w = w_1 \ldots w_n \in S_n$ is Grassmannian with $des(w) = \{k\}$. Then $w_1 \ldots w_k$ and $w_{k+1} \ldots w_n$ are increasing sequences. We index the columns of $\mathrm{Tab}(a)$ by $w_k, w_{k-1}, \ldots, w_1$ and the rows by $w_{k+1}, \ldots, w_n$. Inversions in $w$ are of the form $(w_j, w_i)$ where $i \leq k$ and $j > k$. Each $a_l$ swaps some such $w_i$ and $w_j$, and we label the $(i, j)$-entry of $\mathrm{Tab}(a)$ by $p + 1 - l$.

The extension to isotropic Grassmannian words is straightforward. Let $w = w_{\bar{n}} \ldots w_{\bar{1}} w_1 \ldots w_n \in B_{n-1}$ be isotropic Grassmannian. Then $w_{\bar{n}} \ldots w_{\bar{1}}$ and $w_1 \ldots w_n$ are increasing sequences. We apply Tab as above. Note the entries indexed by $(w_{\bar{i}}, w_j)$ and $(w_{\bar{j}}, w_i)$ correspond to the same swap, hence receive the same label. Since they are symmetric about the diagonal of Tab, we identify them to obtain a shifted standard Young tableau $\mathrm{Tab}_B$. This map is, to our knowledge, a novel bijection between reduced words of isotropic Grassmannians and shifted standard Young tableaux. Examples of Tab and Tab$'$ can be seen in Examples 3.3 and 3.4.

We are now prepared to define the Type A and Type B Little maps. Let $a \in R(w)$ with $w \in A_\infty$ and $(r, s)$ be the last inversion in $w^{-1}$ lexicographically.

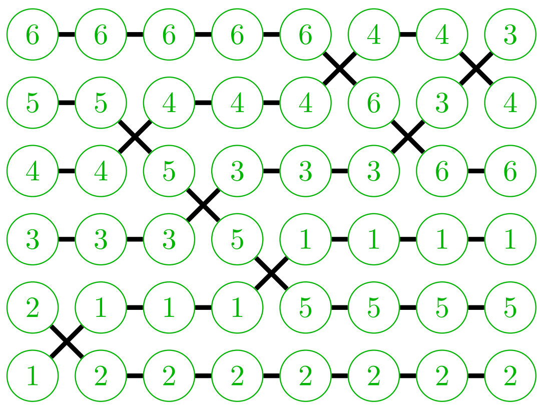
(a) If $a$ is not Grassmannian, set $a := B^-_{(w_s, w_r)}(a)$.

(b) If $a$ is Grassmannian, return $\mathrm{Tab}(a)$ .

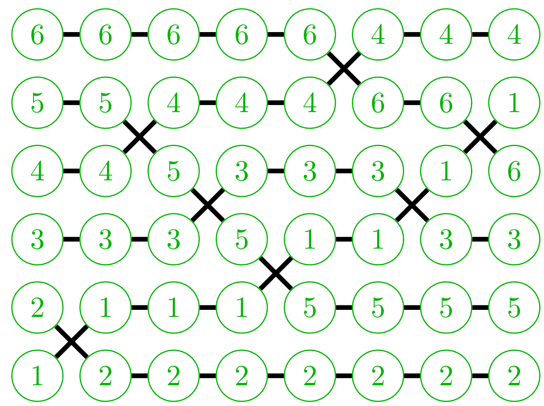
We call the output $\mathrm{LS}(a)$. To obtain the Type B Little map, we use Type B Little bumps and $\mathrm{Tab}_B$ instead and call the output $\mathrm{LS}_B(a)$. Since the transition equations eventually arrive at Grassmannian elements, the algorithm terminates. Note after each Little bump we obtain a reduced word of some new permutation. By recording the sequence of permutations that appear in this way, we can invert each Little bump. Therefore given this sequence of permutations and $\mathrm{LS}(a)$, the Little map is invertible. We demonstrate full examples of the Little map now.

*Example* 3.3. We first demonstrate the Type A Little map. Note $a = 1432545 \in R(346152)$. The initial canonical bump is $B^-_{(3,4)}$. We depict the Little map for $a$.

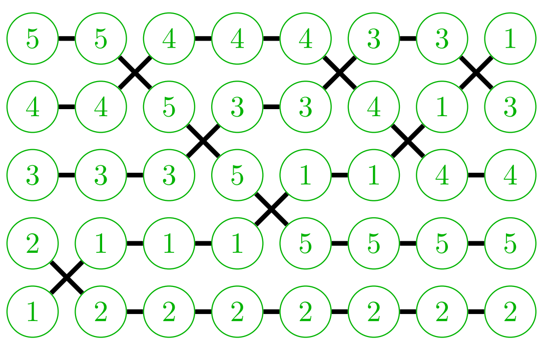
$a$



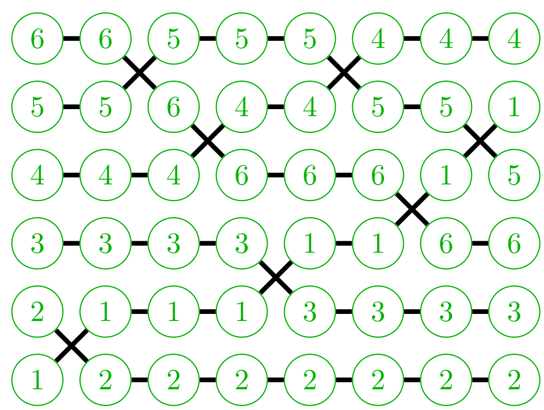

$B^{-}_{(3,4)}(a)$



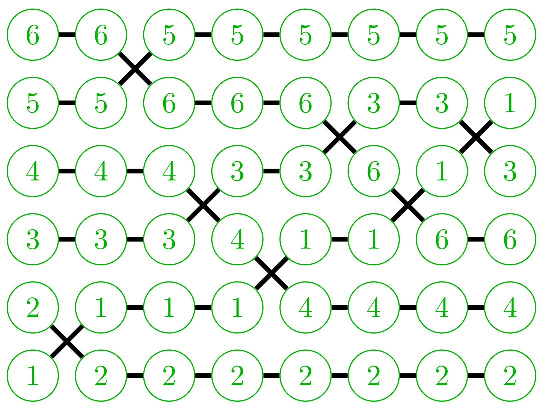

$B^{-}_{(4,6)}B^{-}_{(3,4)}(a)$

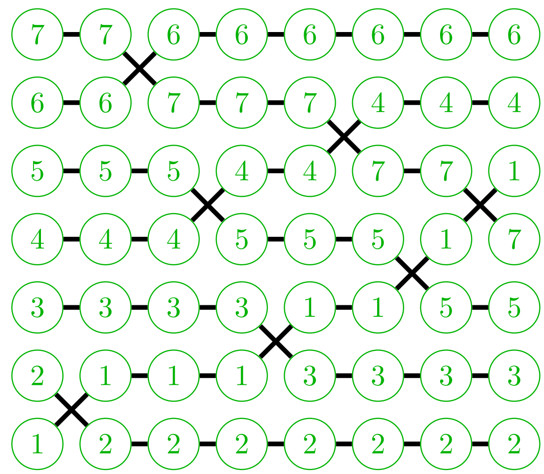



$B^{-}_{(1,5)}B^{-}_{(4,6)}B^{-}_{(3,4)}(a)$



$$B^-_{(4,5)}B^-_{(1,5)}B^-_{(4,6)}B^-_{(3,4)}(a) \qquad\qquad b = B^-_{(1,3)}B^-_{(4,5)}B^-_{(1,5)}B^-_{(4,6)}B^-_{(3,4)}(a)$$




.

Note $b$ is a Grassmannian word. We then compute $\mathrm{LS}(a) = \mathrm{Tab}(b)$ to be

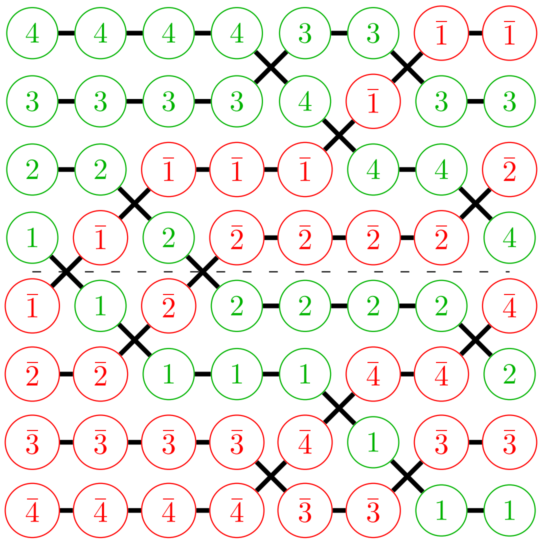
$$\begin{array}{c|c|c|c|} & 1 & 4 & 6 \\ \cline{2-4} 7 & 1 & 3 & 6 \\ \cline{2-4} 5 & 2 & 5 \\ \cline{2-3} 3 & 4 \\ \cline{2-2} 2 & 7 \\ \cline{2-2} \end{array} .$$

Below, we show the image of $a$ under Edelman-Greene insertion for comparison:

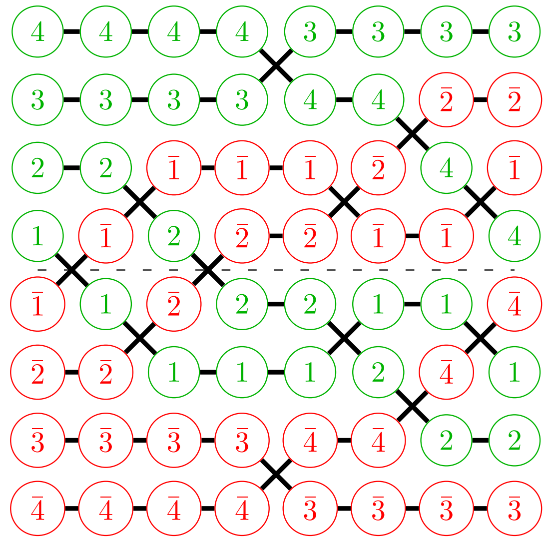
$$\mathrm{P(a)} = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array} \qquad Q(a) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} .$$

*Example* 3.4. We now demonstrate the Type B Little map. Note $a = 0103231 \in R(4\bar{2}3\bar{1})$. The initial canonical bump is $B^-_{(\bar{1},3)}$ since $(w^{-1}_{3}, w^{-1}_{\bar{1}}) = (3,4)$ is the lexicographically largest inversion in $w^{-1}$. We depict the Little map for $a$.

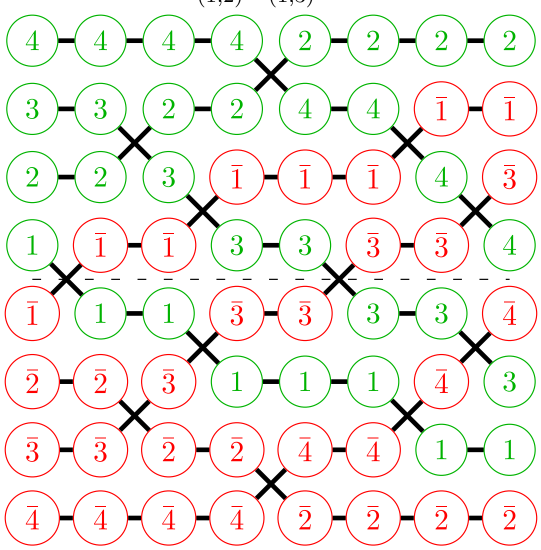
$a$



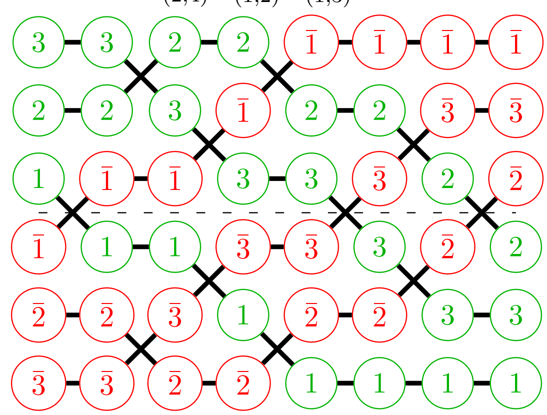
$B^{-}_{(\bar{1},3)}(a)$



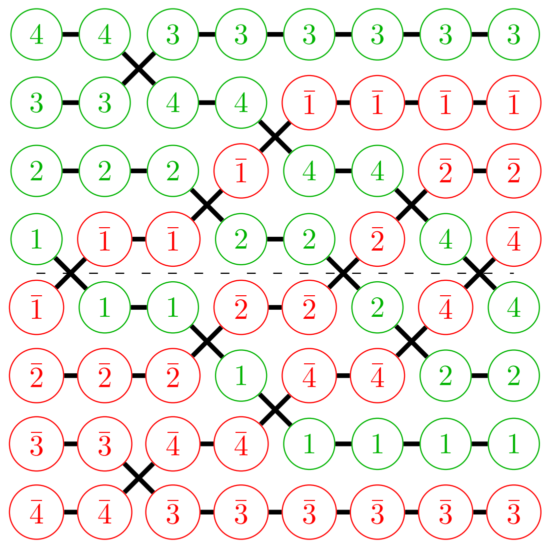
$B^{-}_{(\bar{1},\bar{2})}B^{-}_{(\bar{1},3)}(a)$



$B^{-}_{(2,4)}B^{-}_{(\bar{1},\bar{2})}B^{-}_{(\bar{1},3)}(a)$



$$B^-_{(\bar{3},\bar{2})}B^-_{(2,4)}B^-_{(\bar{1},\bar{2})}B^-_{(\bar{1},3)}(a)$$



.

We then compute $\mathrm{LS}(a) = \mathrm{Tab}(B^-_{(\bar{3},\bar{2})}B^-_{(2,4)}B^-_{(\bar{1},\bar{2})}B^-_{(\bar{1},3)}(a))$ to be

$$\begin{array}{c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{\bar{4}} & \multicolumn{1}{c}{\bar{2}} & \multicolumn{1}{c}{\bar{1}} & \multicolumn{1}{c}{3} \\ \cline{2-5}
4 & 1 & 2 & 4 & 6 \\ \cline{2-5}
2 & \multicolumn{1}{c|}{} & 3 & 5 & \multicolumn{1}{c}{} \\ \cline{3-4}
1 & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{4-4}
\bar{3} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{}
\end{array}\ .$$

Below, we show the image of $a$ under Kraśkiewicz insertion for comparison:

$$P'(a) = \begin{array}{|c|c|c|c|} \hline 3 & 2 & 1 & 0 \\ \hline \multicolumn{1}{c|}{} & 2 & 0 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} \qquad Q'(a) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}\ .$$

# Chapter 4

# Little bumps and Coxeter-Knuth moves

In this chapter, we explore the relationship between Edelman-Greene insertion, Kraśkiewicz insertion and the Little maps. To do so, we first show that Coxeter-Knuth moves and Little bumps commute in a strong sense. This allows us to show that insertion tableaux are invariant under Little bumps. As a consequence, we prove several previously outstanding conjectures about the Little map and present new proofs of many known results within a unified framework. One important consequence of this work is a new algorithm to compute $\mathrm{EG}(w)$ and $\mathrm{Kr}(w)$, the sets of insertion tableaux associated to a permutation or signed permutation.

The results in this section come from two papers. Those regarding the Type A Little map and Edelman-Greene insertion come from [17]. Analogous results for the Type B Little map and Kraśkiewicz insertion are found in [6] or implicit in that work. All conjectures discussed were made for the Type A Little map.

## 4.1 Little bumps and Coxeter-Knuth moves

Little bumps and Coxeter-Knuth moves commute.

**Lemma 4.1.** *Let $a = a_1 \dots a_p$ be a reduced word of $w \in B_\infty$, $\beta_k$ a Type B Coxeter-Knuth move for $a$ and $B^\delta_{(i,j)}$ a Little bump for $w$. Here, if $w \in S_\infty$, we may allow $\beta_k$ to be a Type A Coxeter-Knuth move. Then*

$$B^\delta_{(i,j)}(\beta_k(a)) = \beta_k(B^\delta_{(i,j)}(a)).$$

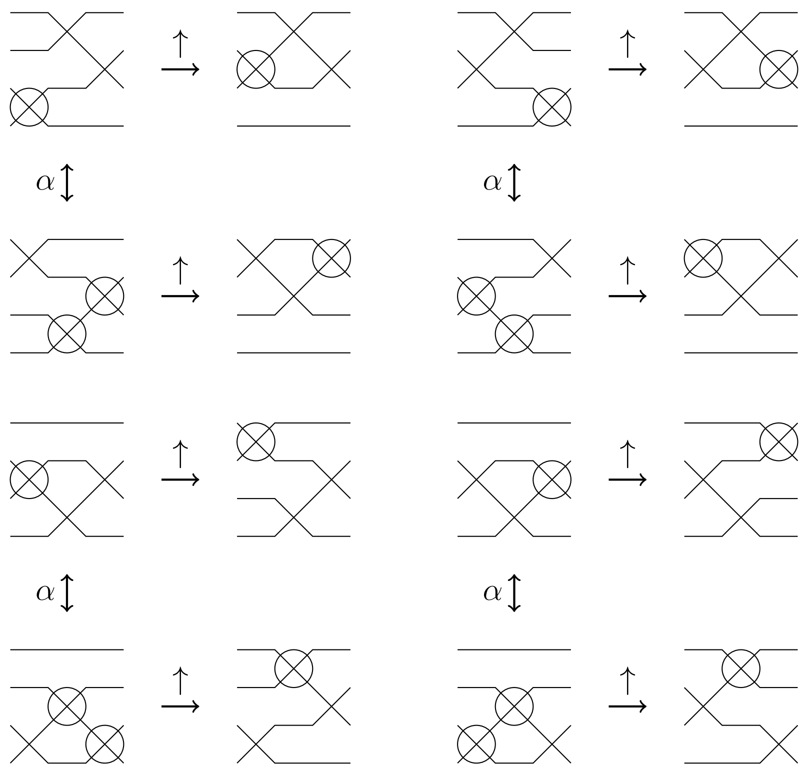
*Proof.* We prove this result when $\beta_k$ is a Type B Coxeter-Knuth move. The proof when it is a Coxeter-Knuth move of Type A is identical, except that the window of $\beta_k$ is of length three instead.

First, observe that a Little bump and the Coxeter-Knuth move $\beta_k$ will only interact if one of the pushes in the bump is applied to an entry in the window of $\beta_k$. The inversions introduced by $a_i$ and $\beta_k(a)_i$ are the same when $a_i$ is not in the window of $\beta_k$. Since $a$ and $\beta_k(a)$ are reduced words of the same permutation, we then see the inversions introduced by $a|_{[k,k+3]}$ and $\beta_k(a)|_{[k,k+3]}$ are the same as well. Therefore if an an entry with index in $[k, k+3]$ is pushed when a Little bump $B^\delta_{(i,j)}$ is applied to $a$, such an index will also be pushed when $B^\delta_{(i,j)}$ is applied to $\beta_k(a)$, though not necessarily the same index. Our argument relies on showing commutation can be reduced to a local check of how the window of $\beta_k$ interacts with $B^\delta_{(i,j)}$. In particular, the result will follow from establishing two properties:

(a) When the pushes applied to entries acted on by a Coxeter-Knuth move resolve, the resulting subwords differ by a Coxeter-Knuth move.

(b) The final push to an entry acted on by the Coxeter-Knuth move has the same

Figure 4.1: The local action of Little bumps on Type A Coxeter-Knuth moves.



effect on $w$ for both $a$ and $\beta_k(a)$, hence would introduce the same defect should the bump continue.

When the entries acted on by a Coxeter-Knuth move differ by two or more, these properties are trivial to confirm. For entries that are closer, the features can be checked for each type of Coxeter-Knuth move either by hand or by computer program. There are four such checks to make for Type A Coxeter-Knuth moves, depicted in Figure 4.1. For each type of Type B Coxeter-Knuth move, there are as many as four checks to make, depending on the initial inversion and whether or not 0 appears in the word. This can be checked by verifying the result for all possible bumps on reduced words in $B_5$ of length 4, which we have done by computer program. $\square$

We are almost prepared to show that recording tableaux are invariant under Little bumps via an argument from canonical form. First, we must define a canonical form. For $\lambda = (\lambda_1, \ldots, \lambda_k)$ a shape, we use the lexicographical ordering of the descent set to induce a partial order on $SYT(\lambda)$. This order has a unique maximal element $U_\lambda$ obtained by filling in rows from left to right, top to bottom. For example,

$$U_{(4,2,1)} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}\ .$$

Note $des(U_{(4,2,1)}) = \{4, 6\}$. In general $des(U_\lambda) = \{\lambda_1, \lambda_1{+}\lambda_2, \ldots\}$. For a shifted shape, we instead use the peak set to obtain $U'_\lambda$ with the lexicographically maximal peak set $peaks(U_\lambda) = \{\lambda_1, \lambda_1{+}\lambda_2, \ldots\}$.

**Theorem 4.1.** *(a) Let $a = a_1 \ldots a_p$ be a reduced word of $w \in S_\infty$, $B^\delta_{(i,j)}$ a Little bump for $w$ and $b = B^\delta_{(i,j)}(a)$. Then*

$$Q(a) = Q(b).$$

*(b) Let $a = a_1 \ldots a_p$ be a reduced word of $w \in B_\infty$, $B^\delta_{(i,j)}$ a Little bump for $w$ and $b = B^\delta_{(i,j)}(a)$. Then*

$$Q'(a) = Q'(b).$$

*Proof.* We first show that $Q(a)$ and $Q(b)$ have the same shape. By Lemma 2.1 and Lemma 2.3,

$$des(Q(a)) = des(a) = des(b) = des(Q(b)).$$

Let $a'$ and $b'$ be the reduced words with maximal peak sets in the Coxeter-Knuth class of $a$ and $b$, respectively. Applying Lemma 4.1, we may assume that $a = a'$.

The shape of $Q(a') = U_\lambda$ is determined by its peak set. Hence, the shape of $Q(b)$ must be at least as large as the shape of $Q(a)$ in dominance order. By assuming that $b = b'$, we can conclude the converse. Therefore $Q(a)$ and $Q(b)$ have the same shape $\lambda$. Furthermore, $Q(a') = U_\lambda = Q(b')$.

We now proceed to showing that $Q(a) = Q(b)$. By Theorem 2.4, there exists a sequence of Coxeter-Knuth moves $A = \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}$ such that $A(a') = a$. From Proposition 2.2, we see

$$Q(a) = Q(A(a')) = h_{i_1}\ldots h_{i_k}(U_\lambda).$$

By Lemma 4.1, $A$ is also a sequence of Coxeter-Knuth moves on $b$, so

$$Q(b) = Q(A(b')) = h_{i_1}\ldots h_{i_k}(U_\lambda),$$

from which we conclude that $Q(a) = Q(b) = Q(B^{\delta}_{(i,j)}(a))$.

The shifted case proceeds identically, using peak sets instead of descent sets.

□

## 4.2 Insertion and the Tab maps

We now show that the Tab maps are alternative characterizations of insertion algorithms. First, we observe a hereditary property of Grassmannians.

**Lemma 4.2.** *Let $a$ be a reduced word.*

*(a) If $a$ is a Grassmannian word, then $a_i \ldots a_p$ is also a Grassmannian word.*

(b) *If $a$ is an isotropic Grassmannian word, then $a_i \dots a_p$ is also an isotropic Grassmannian word.*

*Proof.* The proof proceeds by induction on $i$. Since $w$ is Grassmannian, we see $w$ is comprised of two increasing sequences $x = x_1 \dots x_k$ and $y = y_1 \dots y_l$ with sole descent $x_k > y_1$. Clearly the result holds for $i = 1$. Assume the result holds for $i = k$. Then $s_{a_k}$ swaps the values $a_k$ and $a_k + 1$ in $s_{a_k} s_{a_{k+1}} \dots s_{a_p}$. Since they have swapped, we see $a_k$ and $a_k + 1$ are in $y$ and $x$, respectively. Because they are next to each other, upon swapping them the resulting sequences $x'$ and $y'$ are still increasing. Therefore the resulting permutation is Grassmannian.

□

Next, we study the effect Coxeter moves have on reduced words. In doing so, we make an important observation. Grassmannian permutations avoid the pattern 321. Therefore, their reduced words cannot have the braid relation, since such instances introduce a 321 pattern. Hence all Coxeter relations on Grassmannian words are commuting. By construction, for two Grassmannian words $a = a_1 \dots a_p$ and $b = b_1 \dots b_p$ differing by a Coxeter move, we see $b = a_1 \dots a_{i-1} a_{i+1} a_i a_{i+2} \dots a_p$ and $\mathrm{Tab}(b) = t_{p-i-1,p-i}\mathrm{Tab}(a)$. Note this holds for isotropic Grassmannian words as well.

We now show that Coxeter-Knuth moves have the same effect.

**Lemma 4.3.** *Let $a = a_1 \dots a_p$ be a Grassmannian word and $b = \gamma(a)$ where $\gamma$ is a Coxeter-Knuth move acting non-trivially on $a$. Then $b = a_1 \dots a_{i-1} a_{i+1} a_i a_{i+2} \dots a_p$ and the recording tableaux of $a$ and $b$ differ by the toggle $t_{p-i-1,p-i}$.*

*Proof.* Since Coxeter-Knuth moves are restrictions of Coxeter relations, we see $\gamma$ must act as a transposition. Let $S$ and $T$ be the recording tableaux of $a$ and $b$, respectively.

By Proposition 2.2, we see $S$ and $T$ differ by an elementary dual equivalence, which acts on $S$ as a toggle. To see this toggle is $t_{p-i-1,p-i}$, note by Lemma 4.2 that $a_i a_{i+1} \dots a_p$ and $b_i b_{i+1} \dots b_p$ are reduced words of the same Grassmannian permutation $w$. Therefore $S|_{[p-i]}$ and $S|_{[p-i]}$ have the same shape and differ by a toggle, while $S|_{[p-i-2]} = T|_{[p-i-2]}$. Thus $T = t_{p-i-1,p-i}S$.

□

Additionally, we can deduce that Edelman-Greene insertion coincides with RSK for Grassmannian words since the presence of a special bump indicates a 321 pattern in $w$. Similarly, we see that Kra'skiewicz insertion coincides with mixed shifted insertion for isotropic Grassmannian words. This observation allows us to apply Greene's theorem and its shifted analogue to our insertion algorithms. A sequence of numbers is said to *$k$-decreasing* if it can be partitioned into $k$ disjoint decreasing subsequences, or equivalently if it has no increasing sequence of length $k$+1. In [15], Greene proved the following theorem:

**Theorem 4.2** (Greene's Theorem)**.** *Let $a = a_1 \dots a_p$ such that $P_{RSK}(a)$ has shape $\lambda = (\lambda_1, \dots, \lambda_m)$, and let $I_k(a)$ be length of the longest $k$-decreasing subsequence of $w$. Then for all $k$*

$$I_k(a) = \lambda_1 + \cdots + \lambda_k.$$

This statement is different from usual since insertion takes place from right to left.

An analogue for shifted insertion was proved by Serrano in [18], extending Corollary 5.2 in [36]. Similarly, a sequence is *$k$-unimodal* if it can be partitioned into $k$ unimodal subsequences where each pair of subsequences has at most one entry in common and no entry appears in more than two of the subsequences.

**Theorem 4.3.** *Let $a = a_1 \dots a_p$ be a word such that $P'_{mix}(a)$ has shifted shape $\lambda = (\lambda_1, \dots, \lambda_m)$, and let $I'_k(a)$ be length of the longest $k$-unimodal subsequence of $w$. Then for all $k$*

$$I_k(a) = \lambda_1 + \cdots + \lambda_k + \binom{k}{2}.$$

On the lefthand side the $\binom{k}{2}$ term can be interpreted as counting the intersections, while on the right hand side it represents the initial empty entries of $P'(w)$.

**Theorem 4.4.** *Let $a$ be a reduced word.*

*(a) If $a$ is a Grassmannian word, then $Q(a) = Tab(a)$.*

*(b) If $a$ is an isotropic Grassmannian word, then $Q'(a) = Tab'(a)$.*

*Proof.* Let $w = x_1 \dots x_k y_1 \dots y_{n-k}$ be Grassmannian. From Lemma 4.3, the result holds if we can demonstrate a reduced word $b$ of $w$ whose recording tableaux is $\text{Tab}(a)$. Let $b$ be the inverse of $U_\lambda$ under Tab, so that $\text{Tab}(b) = U_\lambda$. Then $b_{p-\lambda_1} \dots b_p$ are all of the swaps in $b$ featuring $y_1$, hence must be decreasing. By Greene's Theorem, they must then insert into the same row, filling it. Similarly, $b_{p-\lambda_1-\lambda_2} \dots b_{p-\lambda_1-1}$ are all of the swaps in $b$ featuring $y_2$, hence must be decreasing as well. Therefore $b_{p-\lambda_1-\lambda_2} \dots b_p$ is 2-decreasing, so by Greene's Theorem we see $b_{p-\lambda_1-\lambda_2} \dots b_{p-\lambda_1-1}$ must insert into the second row, filling it. Continuing this process, we see $Q(b) = U_\lambda$, completing our proof.

For the isotropic Grassmannian case, we apply identical reasoning. Here $b$ will be the inverse of $U'_\lambda$ under $\text{Tab}'$, and we will concern ourselves with $k$-unimodal sequences and the shifted analogue of Greene's Theorem.

$\square$

## 4.3 Results

Using Theorem 4.1, we can prove many previously outstanding conjectures about the Little map. The first such result we present is a conjecture of Thomas Lam's, originally stated as Conjecture 2.5 in [24]. We say two reduced words $a$ and $b$ *communicate* if there exists a sequence of Little bumps $B^{\delta_1}_{(i_1,j_1)}, \ldots, B^{\delta_k}_{(i_k,j_k)}$ such that

$$b = B^{\delta_1}_{(i_1,j_1)}, \ldots, B^{\delta_k}_{(i_k,j_k)} a.$$

**Theorem 4.5.** *Two reduced words $a$ and $b$ communicate if and only if their recording tableaux are the same.*

*Proof.* If $a$ and $b$ communicate, by Theorem 4.1 they must have the same recording tableaux.

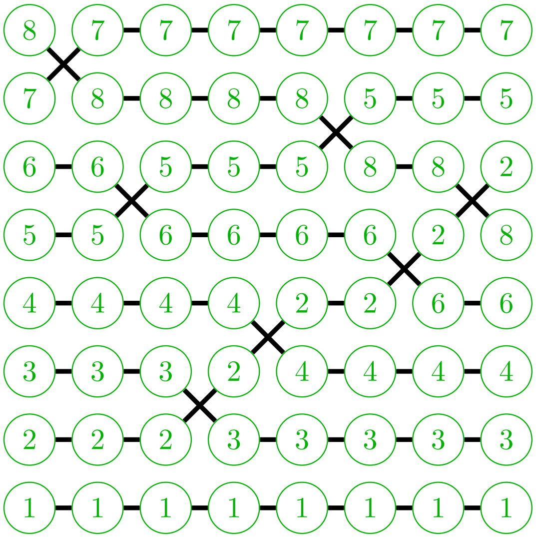
Let $a$ be a reduced word with recording tableau $Q$ of shape $\lambda$. By applying the canonical sequence of Little bumps used in the Little map to $a$, we see $a$ communicates with some word $c \in R(v)$ where $v$ is Grassmannian.. Since they communicate, the recording tableau of $c$ is also $Q$. By 4.4, $R(v)$ is in bijection with $SYT(\lambda)$. Therefore $c$ is the unique member of $R(v)$ with recording tableau $Q$, so $c$ is determined by $Q$. In the shifted case, since $v$ is determined by $Q$, we see every reduced word $a$ with recording tableau communicates with $c$.

In the the unshifted case, there are multiple Grassmannian permutations corresponding the same shape. Such permutations differ by some number of initial fixed points. To finish our proof, we demonstrate a sequence of Little bumps that, when applied to $c$, will remove an initial fixed point from $v$. Let $v = x_1 \ldots x_k y_1 \ldots y_{n-k}$ be a Grassmannian permutation with $x_k y_1$ its sole descent. We construct our sequence of Little bumps by initiating a Little bump at the last swap featuring each $y_j$, beginning with $y_1$. See Example 4.1 for a demonstration. If $v$ has initial fixed points, this sequence will decrement each entry of $c$, removing an initial fixed point.

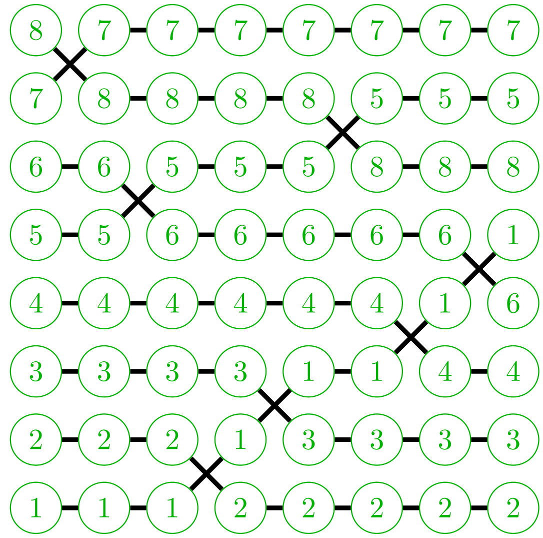
We now verify that our sequence works as described. First, we must verify that the swap locations at which we begin a Little bump are valid choices, that is that removing that swap from $c$ leaves a reduced word. To see this, note that the first such swap chosen is the swap between $x_k$ and $y_1$, the last swap in $a$. This bump will decrement every entry in the trajectory of $y_1$. After the first Little bump, the second swap chosen is the last in the trajectory of $y_2$. Since the trajectories of all $y_j$ with $j > 2$ are unaffected by the initial Little bump, this is the last swap for both $y_2$ and $x_k$, so removing it leaves a reduced word. This bump will decrement every entry in the trajectory of $y_2$. Note because we have already decremented the swaps in the trajectory of $y_1$ and these trajectories were initially disjoint, they will remain disjoint after the second Little bump. Applying this line of reasoning inductively, we see that each Little bump in the sequence is a valid Little bump which decrements every entry of each trajectory. This completes our proof. $\square$

*Example* 4.1. We remove the initial fixed point of the Grassmannian word $w = 7523645 \in R(13468257)$ via the sequence of Little bumps $B^-_{(5,6)}, B^-_{(6,7)}$ and $B^-_{(7,8)}$, as determined in the proof of Theorem 4.5.

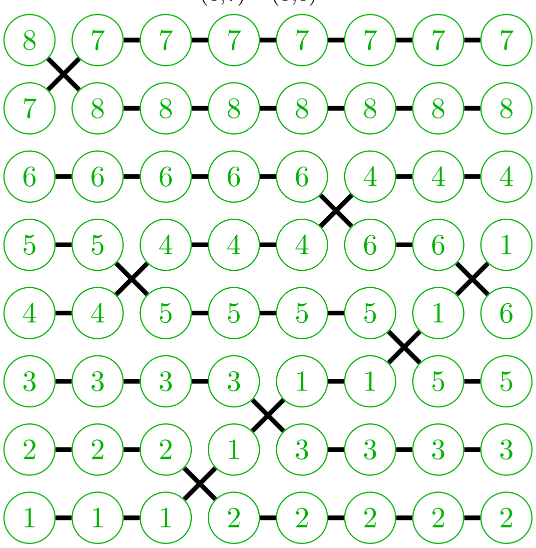
$a$



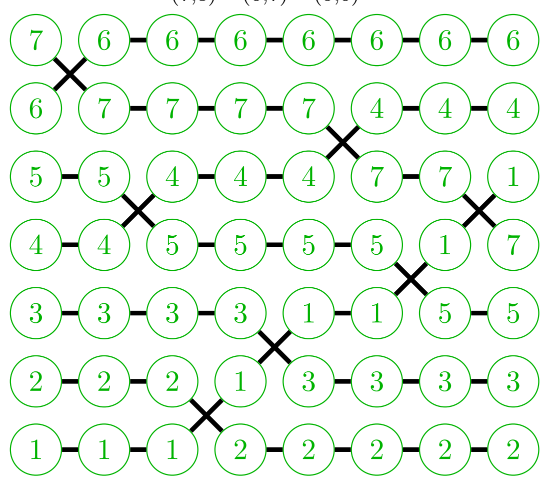
$B^-_{(5,6)}(a)$



$B^-_{(6,7)}B^-_{(5,6)}(a)$



$B^-_{(7,8)}B^-_{(6,7)}B^-_{(5,6)}(a)$



There are many sequences that produce the same outcome.

In his original paper, which also appeared as the appendix of [13], Little made several conjectures about the Little map. The most influential of these is Conjecture 4.3.2 of [13]. For $a \in R(w_0)$, it asserts that $\mathrm{LS}(a) = Q(a)$. This result is true in much greater generality.

**Theorem 4.6.** *The recording tableau, viewed as a map on reduced words, is the same as the Little map.*

*Proof.* The result follows from Theorems 4.1 and 4.4.

□

The following corollary appeared as Conjecture 11 in [27] and Conjecture 4.3.3 in the appendix of [13]. It follows from Theorem 4.5.

**Corollary 4.1.** *Let $a$ be a reduced word and $B^{\delta_1}_{(i_1,j_1)}B^{\delta_2}_{(i_2,j_2)}\dots B^{\delta_k}_{(i_k,j_k)}$ a sequence of Little bumps such that*

$$b = B^{\delta_1}_{(i_1,j_1)}B^{\delta_2}_{(i_2,j_2)}\dots B^{\delta_k}_{(i_k,j_k)}a$$

*is a Grassmannian word. Then $Tab(b) = LS(a)$.*

In [28], Little showed that the Little map coincides with RSK for certain reduced words. More specifically, for $w = w_1 \dots w_p$ a permutation, define the word $a_w = (2w_1 - 1)\dots(2w_n - 1)$. This is a reduced word of $v = 2143\dots(2n)(2n-1)$. Little demonstrated a sequence of Little bumps relating $a_w$ to a Grassmannian word $b_w$ and showed $\mathrm{Tab}(b_w) = Q(w)$. He also showed that $P(a_w) = \mathrm{Tab}(b_{w^{-1}})$ using the well-known fact that $Q(w) = P(w^{-1})$ for $w$ a permutation. We recover these results immediately from Theorem 4.5 and Theorem 4.4. Indeed, from this perspective Little's result is obvious, so there is no need to show an analogous results for Kraśkiewicz insertion. However, this process does highlight an interesting phenomenon.

**Corollary 4.2.** *Let $w \in B_\infty$ and $a \in R(w)$. For any set $S \subset \mathbb{N}$, there exists a permutation of the elements of $S$ that communicates with $a$.*

Another important consequence of Theorem 4.1 is a new algorithm for computing the sets $\mathrm{EG}(w)$ and $\mathrm{Kr}(w)$. Let $w \in B_\infty$.

**Algorithm 4.1.** *to compute EG$(w)$ or Kr$(w)$:*

*(a) Compute the (Type B) Lascoux-Schützenberger tree for $w$.*

*(b) For each leaf of the tree, pick a representative reduced word, e.g. the reading word.*

*(c) Apply inverse Little bumps along the tree to each representative until it is a reduced word of $w$. Such bumps are initiated at the swap introducing the inversion $(w_r, w_i)$, following the notation used to define the Lascoux-Schützenberger tree.*

*(d) Compute the insertion tableau for each representative. The set of outcomes will be EG$(w)$ (or Kr$(w)$).*

In [33], Reiner and Shimozono introduced another algorithm for computing these tableaux. However, this algorithm is substantially harder to implement on a computer and appears to be less computationally efficient.

## 4.4 Little bumps: an ahistorical perspective

This section explores the analogy between Little bumps and jeu de taquin. Our objective in doing so is to develop an approach to enumerating reduced words viewing Little bumps as the starting point. From this perspective, we can reconstruct Stanley symmetric functions, the transition equations and most of Edelman-Greene insertion.

We begin by outlining the role of jeu de taquin in RSK. Each permutation $w \in S_n$ can be viewed as a tableau $T(w)$ with shape $\lambda_0^n \setminus \lambda_0^{n-1}$ and reading word $w$. By applying jeu de taquin slides, we obtain $Q(w)$ as the unique normal shape that is dual equivalent to $T(w)$. Elementary dual equivalences, when applied to $T(w)$, are dual

Knuth transformations. Since $P(w) = Q(w^{-1})$, we can compute $(P(w), Q(w))$ via jeu de taquin without using the insertion algorithm. These results also hold for mixed shifted insertion. This duality also allows us to express Knuth transformations as elementary dual equivalences. By using dual equivalence graphs, we can the construct the $(Q-)$ Schur function associated with the normal shape that is dual equivalent to $T(w)$.

The parallels with Little bumps and jeu de taquin are clear. Theorem 4.6 shows that the recording tableau is the output of the Little map. By tracing the image of elementary dual equivalences under Little bumps, as done in the proof of Lemma 4.1, we can construct the list of Coxeter-Knuth moves. This perspective clearly demonstrates that Coxeter-Knuth graphs are dual equivalence graphs. We can define the Stanley symmetric functions as the symmetric functions arising from Coxeter-Knuth graphs. The Lascoux-Schützenberger and Billey transition equations follow from Theorem 3.1. Using Little bumps as our starting point, we have reconstructed all of the previously discussed technology for enumerating reduced decompositions except for the insertion tableaux.

# Chapter 5

# Signed Dual Equivalence Graphs

In Chapter 4 we introduced dual equivalence graphs and showed that Coxeter-Knuth graphs are dual equivalence graphs. In this chapter, we revisit shifted dual equivalence graphs and characterize them by their local properties. This characterization mirrors Roberts' approach in [34], which is a refinement of Assaf's axioms in [3]. This work appears in joint work with Sara Billey, Austin Roberts and Benjamin Young [6]. Independently, Assaf has studied shifted dual equivalence graphs in [2].

We begin by redefining shifted dual equivalence graphs. These are the graphs induced on the set of standard shifted Young tableaux by shifted dual equivalence. As an application, we outline a simple proof that the product of $Q$-Schur functions is $Q$-Schur positive. We then characterize them in terms of a pair of local properties called the Commuting Property and the Locally Standard Property.

## 5.1 Shifted dual equivalence graphs

We begin by revisiting the definition of standard shifted dual equivalence graphs, which form the components of shifted dual equivalence graphs. Recall for a strict partition $\lambda \vdash n$ that the shifted dual equivalences $h_i$ act as an involution on $ShSYT(\lambda)$, the standard shifted tableaux of shape $\lambda$.

**Definition 5.1.** Let $\lambda \vdash n$ be a strict partition. The *standard shifted dual equivalence graph* for $\lambda$, denoted

$$\mathcal{SG}_\lambda = (V, \sigma, E_1 \cup \ldots \cup E_{n-3}),$$

has vertex set $V = ShSYT(\lambda)$ and the labeled edge sets $E_i$ for $1 \leq i \leq n-3$, with $T \sim_i U$ if $U = h_i(T)$ and $T \neq U$. Recall from Section 1.3 that that every tableau $T \in ShSYT(\lambda)$ has a peak set, denoted $peaks(T)$, and associated signature $\sigma(T)$. Here $\sigma(T) \in \{+,-\}^n$ with $\sigma_i(T) = +$ if and only if $i$ is a peak in $T$. Recall peaks never occur in positions 1 or $n$ and never occur consecutively. Any subset of $[n]$ that satisfies these properties is the peak set of some tableau, and is called an *admissible peak set*.

In Figure 5.1, all of the standard shifted dual equivalence graphs of degree 6 are drawn and labeled with their signatures. We omit $\sigma_1$ and $\sigma_6$ since 1 and 6 can never be in an admissible peak set. From this figure we see that standard shifted dual equivalence graphs can have triple edges, which cannot occur for ordinary dual equivalence graphs. Note for $v \sim_i w$ that $\sigma(v)_{[i+1,i+2]} = -\sigma(w)_{[i+1,i+2]}$. Furthermore, the labels of each edge can be determined entirely from the peak sets. This fact will be used often in the proofs that follow. From this figure, we also can determine all of the possible standard shifted dual equivalence graphs for $n = 1, 2, 3, 4, 5$ by fixing

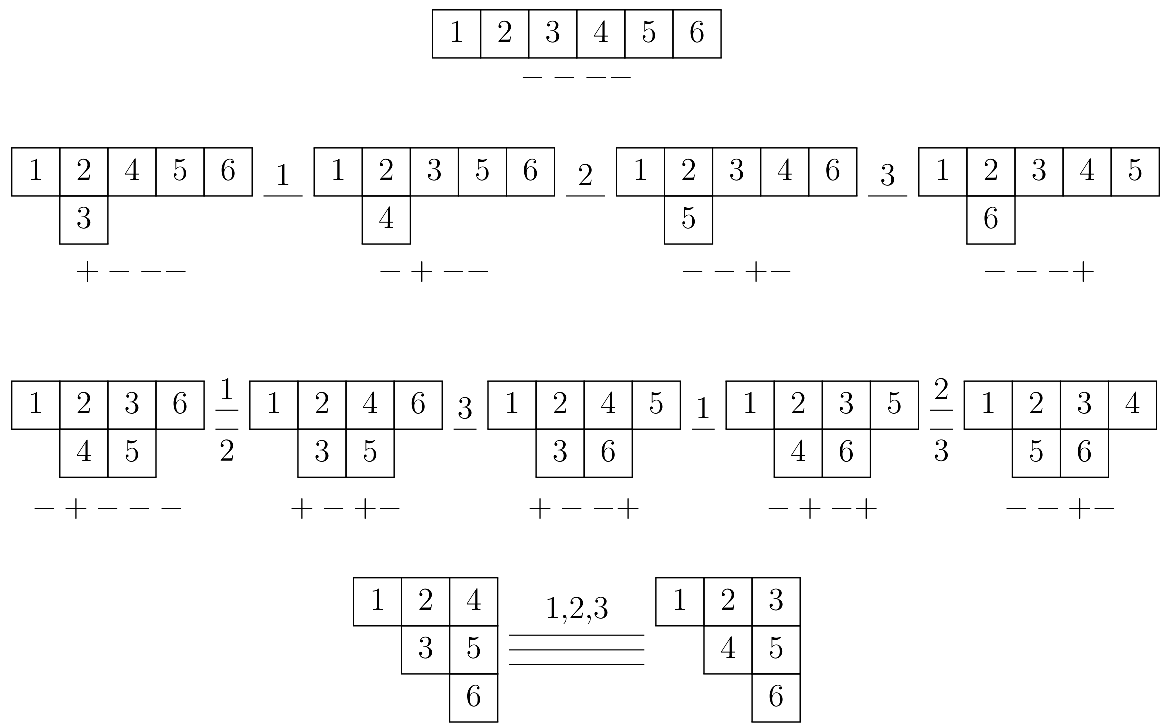



Figure 5.1: The standard shifted dual equivalence graphs of degree 6 omitting $\sigma_1$ and $\sigma_6$.

the position of higher values.

The standard shifted dual equivalence graphs have several nice properties on par with the dual equivalence graphs and Coxeter-Knuth graphs of type $A$. By Theorem 2.1, each $\mathcal{SG}_\lambda$ is connected. Additionally, Theorem 1.5 shows that the Schur $Q$-function $Q_\lambda$ is the generating function for the sum of peak quasisymmetric functions associated to the labels on the vertices. Recall from Chapter 4 that the lexicographically largest peak set for all standard shifted tableaux of a fixed shape $\lambda$ is given by the unique tableau $U_\lambda$. Thus, the shape $\lambda$ can be recovered from the multiset of signatures.

Each standard shifted dual equivalence graph is an example of the following more general type of graph.

**Definition 5.2.** Given two finite sets $C$ and $S$, an *$S$-signed, $C$-colored graph* consists of the following data:

(a) a finite vertex set $V$,

(b) a signature function $\sigma : V \to \{+,-\}^{|S|}$ associating a subset of $S$ to each vertex,

(c) a collection $E_i$ of unordered pairs of distinct vertices in $V$ for each $i \in C$.

A signed colored graph is denoted $\mathcal{G} = (V, \sigma, E)$ where $E = \cup_{i\in C} E_i$. We say that $\mathcal{G}$ has *shifted degree* $n$ if $C = [n-3]$, $S = [n]$ and $\sigma(v)$ is an admissible peak set for each $v \in V$. The signature $\sigma(v)$ is encoded by a sequence in $\{+,-\}^n$ where $\sigma(v)_i = +$ if and only if $i \in \sigma(v)$. A *morphism* between signed colored graphs are functions on vertex sets that preserve edge labels and signatures. An *isomorphism* is a bijective morphism whose inverse is also a morphism.

**Definition 5.3.** A signed colored graph $\mathcal{G}$ is a *shifted dual equivalence graph* (SDEG) if it is isomorphic to a disjoint union of standard shifted dual equivalence graphs.

The isomorphism type of a connected SDEG is determined uniquely by a strict partition.

**Lemma 5.1.** *Let $\mathcal{SG}_\lambda$ and $\mathcal{SG}_\mu$ be any two standard shifted dual equivalence graphs. If $\varphi\colon \mathcal{SG}_\lambda \to \mathcal{SG}_\mu$ is an isomorphism, then $\lambda = \mu$ and $\varphi$ is the identity map.*

*Proof.* Let $\varphi : \mathcal{SG}_\lambda \to \mathcal{SG}_\mu$ be an isomorphism. Then the vertices of $\mathcal{SG}_\lambda$ and $\mathcal{SG}_\mu$ have the same multisets of associated peak sets. By looking at the unique lexicographically maximal peak set in both, it follows that $U_\lambda = U_\mu$. In particular, $\lambda = \mu$. Thus, $\varphi$ is an automorphism sending $U_\lambda$ to itself. Since elementary dual equivalences define the same edges in $\mathcal{SG}_\lambda$ and $\mathcal{SG}_\mu$ and the graphs are connected, we see that $\varphi$ acts as the identity map. □

Theorem 2.10 states that Coxeter-Knuth graphs are dual equivalence graphs and that standard shifted dual equivalence graphs are Coxeter-Knuth graphs.

**Definition 5.4.** Given a signed colored graph $\mathcal{G} = (V, \sigma, E)$ of shifted degree $n$ and an interval of nonnegative integers $I = [a, b] \subset [n]$, let

$$\mathcal{G}^I = (V, \sigma, E_a \cup E_{a+1} \cup \cdots \cup E_{b-3})$$

denote the subgraph of $\mathcal{G}$ using only the $i$-edges for $a \leq i \leq b-3$. Also define the *restriction of* $\mathcal{G}$ *to* $I$, to be the signed colored graph

$$\mathcal{G}|_I = (V, \sigma', E')$$

with

(a) $\sigma'(v) = \{s - a + 1 \mid s \in \sigma(v) \cap (a, b)\}$,

(b) $E'_i = E_{a+i-1}$ when $i \in [|I| - 3]$.

Notice that the vertex sets of $\mathcal{G}$, $\mathcal{G}^I$ and $\mathcal{G}|_I$ are the same. If $\mathcal{G}$ is a signed colored graph with shifted degree $n$ and $I = [a, b]$ then $\mathcal{G}|_I$ will have shifted degree at most $|I|$. It could be strictly less than $|I|$ if $n < b$.

## 5.2 Identifying shifted dual equivalence graphs

Theorem 2.9 characterizes ordinary dual equivalence graphs in terms of two local properties. The analogous properties for shifted dual equivalence graphs are the following:

(a) **Locally Standard**: If $I$ is an interval of positive integers with $|I| \leq 9$, then each component of $\mathcal{G}|_I$ is isomorphic to a standard shifted dual equivalence graph of degree up to $|I|$.

(b) **Commuting**: If $(u, v) \in E_i$ and $(u, w) \in E_j$ then there exists a vertex $y \in V$ such that $(v, y) \in E_j$ and $(w, y) \in E_i$. Thus the components of $(V, E_i \cup E_j)$ for $|i - j| > 4$ are commuting diamonds.

Recall for ordinary shifted dual equivalence graphs that Locally Standard Property extended only to $l \leq 6$ and the Commuting Property required that $|i-j| > 3$. We now set out to prove that these properties characterize shifted dual equivalence graphs.

**Theorem 5.1.** *Let $\mathcal{G}$ be a signed colored graph satisfying the Commuting Property and the Locally Standard Property. Then $\mathcal{G}$ is a shifted dual equivalence graph.*

The proof of this theorem is quite involved, requiring many technical lemmas.

**Lemma 5.2.** *For any standard shifted dual equivalence graph $\mathcal{SG}_\lambda$, the Locally Standard Property and the Commuting Property hold. In fact, $\mathcal{SG}_\lambda|_{[I]}$ is an SDEG for all intervals $I$.*

*Proof.* Let $\mathcal{SG}_\lambda$ be a standard shifted dual equivalence graph with $\lambda \vdash n$. We can restrict any $T \in ShSYT(\lambda)$ to the values in $I$ to form a skew shifted tableau. All the data for $\mathcal{SG}_\lambda|_I$ will still be determined. Since jeu de taquin slides commute with the $h_i$'s, the isomorphism from $\mathcal{SG}_\lambda|_I$ to a union of standard shifted dual equivalence graph is given by restriction and repeated application of the jeu de taquin operator $\Delta$ defined in Chapter 2.

The Commuting Property holds since the action of $h_i$ depends only on the positions of the values in $[i, i+4]$. Therefore, $h_i$ and $h_j$ commute provided $|i-j| > 4$. $\square$

Lemma 5.2 can also be proved by appealing to the fact that $\mathcal{SG}_\lambda$ is isomorphic to a Coxeter-Knuth graph. We know the $\beta_i$'s satisfy the Commuting Property. Furthermore, restriction on a Coxeter-Knuth graph gives rise to an isomorphism with another Coxeter-Knuth graph since every consecutive subword of a reduced word is again reduced.

**Lemma 5.3.** *Given a strict partition $\lambda$ of size $n$, any two distinct components $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{SG}_\lambda^{[n-1]}$ are connected by an $(n-3)$-edge in $\mathcal{SG}_\lambda$. In particular, any two vertices in $\mathcal{SG}_\lambda$ are connected by a path containing at most one $(n-3)$-edge that is not also an $(n-4)$-edge.*

*Proof.* It follows from Theorem 2.1 that $\mathcal{A}$ and $\mathcal{B}$ are characterized by the position of $n$ in their respective shifted tableaux. Suppose $\mathcal{A}$ and $\mathcal{B}$ have $n$ in corner cells $c$ and $d$, respectively, with $c$ is in a lower row than $d$. Then there exist tableaux $S \in \mathcal{A}$ and $T \in \mathcal{B}$ that agree everywhere except in the cells containing $n-1$ and $n$ such that $n-2$ lies between $n$ and $n-1$ in the reading word of $S$ and $n-3$ comes before $n-2$. Then by definition of $h_{n-3}$ we have $h_{n-3}(S) = T$, so $\mathcal{A}$ and $\mathcal{B}$ are connected by an $(n-3)$-edge. This edge cannot be an $(n-4)$-edge since $\mathcal{A}$ and $\mathcal{B}$ are not connected in $\mathcal{SG}_\lambda^{[n-1]}$. $\square$

**Lemma 5.4.** *Let $\mathcal{G} = (V, \sigma, E)$ be a signed colored graph of shifted degree $n$ satisfying the Commuting Property and the local condition that $\mathcal{G}|_{[j,j+5]}$ is a shifted dual equivalence graph for all $1 \leq j \leq n-5$. If $v, w \in V$ are connected by an $i$-edge in $\mathcal{G}$, then $\sigma_k(v) = \sigma_k(w)$ for all $k \notin [i-1, i+4]$.*

*Proof.* The lemma clearly holds for standard SDEGS by the definition of the shifted dual equivalence moves $h_i$ which determine the $i$-edges. For $k = i+5$, the lemma holds since $\mathcal{G}|_{[i,i+5]}$ is a shifted dual equivalence graph. Now assume that $i+5 < k \leq n$. Let $v, w \in V$ such that $v \sim_i w$ and assume $\sigma_j(v) = \sigma_j(w)$ for all $i+5 \leq j < k$ by induction. By the local condition, the vertex $v$ admits an $(k-2)$-edge if and only if $\sigma_{k-1}(v) = +$ or $\sigma_k(v) = +$. These possibilities are mutually exclusive since the signature encodes an admissible peak set. Thus, $\sigma_k(v)$ is determined by $\sigma_{k-1}(v)$ and the presence or absence of an adjacent $(k-1)$-edge. By the Commuting Property $i$-edges and $(k-2)$-edges commute for $k-2-i \geq 4$, so $v$ admits a $(k-2)$-edge if and only if $w$ admits a $(k-2)$-edge. Since $\sigma_{k-1}(v) = \sigma_{k-1}(w)$, we obtain $\sigma_k(v) = \sigma_k(w)$ by the same considerations. By induction we see $\sigma_k(v) = \sigma_k(w)$ for all $i+4 < k \leq n$.

A similar argument works for all $1 \leq k < i-1$, with base case $\sigma_{i-2}(v) = \sigma_{i-2}(w)$ since $\mathcal{G}|_{[i-2,i+3]}$ is a shifted dual equivalence graph. This completes the proof. $\square$

**Lemma 5.5.** *Let $\lambda$ be a strict partition of $n$ and $\mathcal{G} = (V, \sigma, E)$ be a signed colored graph of shifted degree $n$ satisfying the Locally Standard and Commuting Properties. If $\varphi : \mathcal{G} \longrightarrow \mathcal{SG}_\lambda$ is an injective morphism, then it is an isomorphism.*

*Proof.* Let $v \in V$ and say $\varphi(v) = T \in SST(\lambda)$. Since $\varphi$ is signature preserving and $\mathcal{G}$ is Locally Standard, Lemma 5.4 shows that $v$ has an $i$-neighbor in $\mathcal{G}$ if and only if $T$ has an $i$-neighbor in $\mathcal{SG}_\lambda$ and a similar statement holds for each of their neighbors. Furthermore, since $\varphi$ is an injective morphism $(v, w) \in E_i \cap E_j$ if and only if $h_i(T) = h_j(T) = \varphi(w)$. Thus, $\varphi$ induces a bijection from the neighbors of $v$ to the neighbors of $T$ that preserves the presence or absence of $i$-neighbors. In particular, every neighbor of $T$ in $\mathcal{SG}_\lambda$ is in the image of $\varphi$. Since $\mathcal{SG}_\lambda$ is connected, there is a path from $T$ to any other vertex $S$ in $\mathcal{SG}_\lambda$ and by iteration of the argument above we see that $\varphi$ maps some vertex in $V$ to $S$. Hence, $\varphi$ is both injective and surjective on vertices, and the inverse map is also a morphism of signed colored graphs. Thus, $\varphi$ is an isomorphism. □

With Lemma 5.5 in mind, our goal will be to demonstrate the existence of a morphism from any connected signed colored graph satisfying the Locally Standard and Commuting Properties to a standard SDEG and show that this morphism is injective. To do this, we will employ an induction on the degree of the signed colored graphs in questions. The next lemma is an important part of that induction.

**Lemma 5.6.** *Let $\mathcal{G} = (V, \sigma, E_1 \cup \ldots \cup E_{n-2})$ be a signed colored graph of shifted degree $n+1$ that satisfies the following hypotheses:*

*(a) The Commuting Property holds on all of $\mathcal{G}$.*

*(b) Both $\mathcal{G}|_{[n]}$ and $\mathcal{G}|_{[n-6,n+1]}$ are shifted dual equivalence graphs.*

*Let $\mathcal{C}$ be a component of $\mathcal{G}^{[n]}$. The following properties hold:*

*(a) There exists a unique strict partition $\mu$ of degree $n+1$ and a signed colored graph isomorphism $\varphi$ mapping $\mathcal{C}$ to a component of $\mathcal{SG}_\mu^{[n]}$.*

*(b) For every vertex $v \in V(\mathcal{C})$, $v$ has an $(n-2)$-neighbor in $\mathcal{G}$ if and only if $\varphi(v)$ has an $(n-2)$-neighbor in $\mathcal{SG}_\mu$.*

We refer to $\mathcal{SG}_\mu$ in this lemma as the *unique extension* of $\mathcal{C}$ in $\mathcal{G}$. The outline of this proof is based on the proof of Theorem 3.14 in [3], but it uses peak sets in addition to ascent/descent sets for tableaux.

*Proof.* By hypothesis, $\mathcal{C}|_{[n]}$ is isomorphic to $\mathcal{SG}_\lambda$ for some strict partition $\lambda \vdash n$. Therefore we can bijectively label the vertices of $\mathcal{C}$ by standard shifted tableaux of shape $\lambda$ while preserving the restricted signature function $\sigma|_{[n]}$. Since $\mathcal{G}|_{[n-6,n+1]}$ is an SDEG, the lemma is automatically true if $n \leq 7$, so assume $n > 7$.

We partition the vertices of $\mathcal{C}$, or equivalently $SST(\lambda)$, according to the placement of $n-1$ and $n$. Let $D_{ij}$ be the subgraph of $\mathcal{C}$ with $n$ in row $i$ and $n-1$ in row $j$ with edges in $E_1 \cup \ldots \cup E_{n-5}$. Each $D_{ij}$ is connected since its restriction to $[n-2]$ is also isomorphic to a standard SDEG by hypothesis. Similarly, let $D_i$ be the connected subgraph of $\mathcal{C}$ with $n$ in row $i$ edge set $E_1 \cup \ldots \cup E_{n-4}$.

We first show that $\sigma_n$ is constant on $D_{ij}$. By Lemma 5.3, each pair $S, T \in V(D_{ij})$ may be connected by a path using only edges in $E_1 \cup \cdots \cup E_{n-5}$. Then by Lemma 5.4 $\sigma_n$ is constant on $D_{ij}$. The same fact need not hold for the $D_i$. We will show that there is a unique row of $\lambda$ where $n+1$ can be placed that is simultaneously consistent with the signatures for all vertices in all the $D_i$'s. This placement will also be consistent with the existence of $n-2$-edges in $\mathcal{G}$, completing the proof.

We proceed by partitioning the $D_{ij}$ for a fixed $i$ into three types and describing how to extend each type consistently. First, suppose that there is some nonempty $D_{ij}$ with $i > j$. Then $n$ is in a strictly lower row than $n-1$ for all the tableaux labeling vertices of $D_{ij}$. Furthermore there is some tableau $T$ labeling a vertex of $D_{ij}$ such that $n-2$ lies in row weakly above row $j$ making position $n-1$ a peak. This implies $\sigma_n(T) = -$ since peaks cannot be adjacent. Since $\sigma_n$ is constant on $D_{ij}$, we see that

$i > j$ implies $\sigma_n(T) = -$ for all tableaux labeling vertices in $D_{ij}$. Furthermore, any placement of $n{+}1$ will be consistent with the fact that $\sigma_n = -$.

Second, suppose that $i \le j$ and that $\sigma_n(T) = +$ for all $T \in V(D_{ij})$. We would like to add $n{+}1$ to a row strictly above $i$, but we must show this will be consistent with each neighboring component of $D_{ij'}$ in $D_i$. By Lemma 5.3, the component $D_{ij}$ is connected to every other $D_{ij'}$ by an $(n-4)$-edge. Such an edge $e = (T, U) \in V(D_{ij}) \times V(D_{ij'})$ could also be an $(n-3), (n{-}2)$-edge. In this case, we must have $\sigma_{[n-2,n]}(T) = + - +$ and $\sigma_{[n-2,n]}(U) = - + -$, as seen in Figure 5.1. Thus, if $U$ is a vertex in $D_{ij'}$, then position $n{-}1$ must be a peak of $U$ and $i > j'$. Therefore, if $n{+}1$ is added in any row to a tableau $T \in V(D_{ij'})$ it will not create a peak in position $n$. On the other hand, if $D_{ij}$ is connected to another nonempty $D_{ij'}$ by an $(n-4)$-edge that is not also a $(n{-}2)$ edge, then again by Figure 5.1 we see that $\sigma_n(U) = +$ for all $U \in V(D_{ij'})$. Thus, we can consistently extend each vertex in $D_i$ by placing $n{+}1$ in such a way that it creates a descent from $n$ to $n{+}1$. Any row strictly below row $i$ will suffice, provided it results in another shifted shape.

Third, suppose there exists a nonempty $D_{ij}$ such that $i \le j$ and $\sigma_n(T) = -$ for all $T \in V(D_{ij})$. As before $D_{ij}$ is connected to every other $D_{ij'}$ by an $(n-4)$-edge. Assume such an edge $e = (T, U) \in V(D_{ij}) \times V(D_{ij'})$ is part of a triple edge with an $(n-3), (n{-}2)$-edge. In this case, we must have $\sigma_{[n-2,n]}(T) = - + -$ since $\mathcal{G}|_{[n-6,n+1]}$ is an SDEG. Thus, $n{-}1$ is a peak of $T$, but this contradicts the assumption that $i \le j$. Therefore no $(n-4)$-edge containing a vertex in $D_{ij}$ is also an $(n{-}2)$-edge. By observing Figure 5.1, we conclude that $\sigma_n = -$ on all of $D_{ij'}$. In this case, we can consistently extend all tableaux labeling vertices in $D_i$ by placing $n{+}1$ in any row weakly above $n$.

We complete the proof by placing $n{+}1$ in a unique row $m$ consistent with the required ascents and descents in all $D_i$. Let $X$ be the union of all nonempty $D_i$ containing a vertex $T$ with $\sigma_n(T) = +$, and let $Y$ be the union of all nonempty $D_i$ with no such vertex. We have shown that, when placing $n{+}1$, every vertex in $X$ needs a descent from $n$ to $n{+}1$, and every vertex in $Y$ needs an ascent from $n$ to $n{+}1$. To do this, let $m$ be the minimal positive integer such that $i < m$ for all $D_i \in X$. We will show that $Y$ consists of all non-empty $D_i$ with $i \geq m$.

If $X$ is empty, then $m = 1$ and we extend $\lambda$ to a strict partition $\mu$ by adding one box to the first row of $\lambda$. Then $\mathcal{C}|_{[n]}$ is isomorphic to the component of $\mathcal{SG}_\mu|_{[n]}$ with $n{+}1$ fixed in the first row, and the conclusions of the lemma hold.

Assume $X$ is nonempty and let $i < j$ such that $D_i$ and $D_j$ are nonempty with $D_i \in Y$. Since $\mathcal{C}$ is connected, there exists an $(n-3)$-edge $e = (S, T)$ with $S \in D_i$ and $T \in D_j$. By the definition of shifted dual equivalence moves on $\mathcal{SG}_\lambda|_{[n]}$ $e$ must act as the transposition $t_{n,n-1}$ on $S$ and $T$. This implies $S$ has $n{-}1$ in row $j$ and $n$ in row $i$. In this configuration, $n{-}1$ cannot be the position of a peak in $S$. Thus $S$ must have a peak in position $n{-}2$ since it is a vertex of an $(n-3)$-edge. If $\sigma_n(S) = +$, then it would contradict the hypothesis that $D_i \in Y$. Therefore $\sigma_{[n-2,n]}(S) = + - -$, which implies $\sigma_{[n-2,n]}(T) = - + -$. In particular, $S$ and $T$ are not connected by an $(n{-}2)$-edge. We can then conclude that $T$ must have an ascent from $n$ to $n{+}1$. Thus $D_j \in Y$ for all $j > i$.

We conclude that $Y$ consists of all the $D_i$ for all $i \geq m$ and $X$ consists of all $D_i$ for $i < m$. By placing $n{+}1$ may placed in row $m$, we obtain a strict partition $\mu$ such that $\mathcal{C}$ is isomorphic to a connected component of $\mathcal{SG}_\mu^{[n]}$. Moreover, no other choice of row can simultaneously satisfy the required ascents and descents from $n$ to $n{+}1$

for all $D_{ij}$. $\square$

We can also find a *unique lower extension* of a component of $\mathcal{G}|_{[2,n+1]}$ provided similar conditions hold.

**Corollary 5.1.** *Let $\mathcal{G} = (V, \sigma, E_1 \cup \ldots \cup E_{n-2})$ be a signed colored graph of shifted degree $n+1$ satisfyingt the Commuting Property such that both $\mathcal{G}|_{[2,n+1]}$ and $\mathcal{G}|_{[1,7]}$ are shifted dual equivalence graphs. Let $\mathcal{C}$ be a component of $\mathcal{G}^{[2,n+1]}$. Then, there exists a strict partition $\mu$ of degree $n+1$ and a signed colored graph isomorphism $\varphi$ mapping $\mathcal{C}$ to a component of $\mathcal{SG}_\mu^{[2,n+1]}$.*

*Proof.* This follows from Lemma 5.6 provided we reverse all the signatures and the edge labels. Reversing signatures and the edge labels is very natural for Coxeter-Knuth graphs since the reverse of a reduced word is reduced. $\square$

**Lemma 5.7.** *Two shifted standard tableau $T$ and $U$ of shifted shape $\lambda \vdash n$ are in the same component of $\mathcal{SG}_\lambda|_{[2,n]}$ if and only if $\Delta(T)$ and $\Delta(U)$ have the same shape.*

*Proof.* Theorem 2.7 states that $\Delta(T)$ and $\Delta(U)$ have the same shape if an only if $T$ and $U$ are connected by elementary dual equivalences $h_2, \ldots, h_{n-3}$.. This is equivalent to saying that they are in the same component of $\mathcal{SG}_\lambda|_{[2,n]}$ $\square$

**Lemma 5.8.** *Let $\mathcal{G} = (V, \sigma, E_1 \cup \cdots \cup E_{n-3})$ be a connected signed colored graph of shifted degree $n$ satisfying the Commuting Property such that $\mathcal{G}|_{[n-1]}$ and $\mathcal{G}|_{[2,n]}$ are SDEGs. Let $\mathcal{C}$ be any component of $\mathcal{G}|_{[n-1]}$, and let $\mathcal{SG}_\mu$ be the unique extension of $\mathcal{C}$ in $\mathcal{G}$. Let $v \in V(\mathcal{C})$, and let $\mathcal{C}'$ be the component of $v$ in $\mathcal{G}|_{[2,n]}$. Say $\mathcal{SG}_\lambda \cong \mathcal{G}|_{[2,n]}$. If $v$ is mapped to $T \in SST(\mu)$ in $\mathcal{SG}_\mu$, then $v$ is mapped to $\Delta(T)$ in $\mathcal{SG}_\lambda$.*

*Proof.* The proof follows from Lemma 5.6 and Lemma 5.7. $\square$

**Lemma 5.9.** *Let $\mathcal{G} = (V, \sigma, E_1 \cup \cdots \cup E_{n-3})$ be a connected signed colored graph of shifted degree $n > 9$ satisfying the Commuting Property such that $\mathcal{G}|_{[n-1]}$ and $\mathcal{G}|_{[2,n]}$ are SDEGs. Let $\mathcal{C}$ be any component of $\mathcal{G}^{[n-1]}$, and let $\mathcal{SG}_\mu$ be the unique extension of $\mathcal{C}$ in $\mathcal{G}$.*

*(a) An $(n-3)$-edge connects two vertices in $\mathcal{C}$ if and only if the corresponding vertices are connected by an $(n-3)$-edge in $\mathcal{SG}_\mu$.*

*(b) If two $(n-3)$-edges connect the image of $\mathcal{C}$ to the same component in $\mathcal{SG}_\mu^{[n-1]}$, then corresponding edges in $\mathcal{G}$ must also connect $\mathcal{C}$ to the same component in $\mathcal{G}^{[n-1]}$.*

*Proof.* We begin by considering $u, v \in \mathcal{C}$ with $u \sim_{n-3} v$. Since $\mathcal{SG}_\mu$ is the unique extension of $\mathcal{C}$, we can associate tableaux $S, T$ with $u, v$ respectively. We want to show $h_{n-3}(S) = T$. By hypothesis $\mathcal{G}|_{[2,n]}$ is an SDEG. The component $\mathcal{C}'$ of $\mathcal{G}|_{[2,n]}$ containing $u$ is isomorphic to the component of $S$ in $\mathcal{SG}_\mu|_{[2,n]}$ by Lemma 5.6. The vertex $u$ maps to $\Delta(S)$ under this isomorphism. Since $u \sim_{n-3} v$, they are connected by an $n-4$-edge in $\mathcal{C}'$. By Lemma 5.8, the image of $v$ in $\mathcal{SG}_\mu|_{[2,n]}$ is $\Delta(T)$ and $h_{n-4}(\Delta(S)) = \Delta(T)$. Therefore, $h_{n-3}(S) = T$ since every edge in $\mathcal{SG}_\mu|_{[2,n]}$ comes from an edge in $\mathcal{SG}_\mu$ with one higher label.

The previous argument is reversible. That is, given $S, T \in SST(\mu)$ with $n$ in the same cell, if $h_{n-3}(S) = T$ then the vertices $u, v$ in $\mathcal{C}$ mapping to $S, T$ respectively must be connected by an $(n-3)$-edge in $\mathcal{G}$. This proves (a).

Next we prove (b). By Lemma 5.6, we can label the vertices of $\mathcal{C}$ by standard tableaux of shape $\mu$. Let $s, t \in \mathcal{C}$ be labeled by the tableaux $S$ and $T$, respectively. Assume that $s \sim_{n-3} s'$ and $t \sim_{n-3} t'$, but $s', t' \notin \mathcal{C}$. Further assume that both $S$ and $T$ are connected to the same component of $\mathcal{SG}_\mu^{[1,n-1]}$ by $(n-3)$-edges, and that this component is distinct from the component of $T$. Then, $n-1$ and $n$ must be in the same cells of $S$ and $T$, with $n-2$ in some cell between the two in row reading order, and $n-3$ in some cell before that, by the definition of $h_{n-3}$.

If $S$ and $T$ are connected via edges in $E_1 \cup \ldots \cup E_{n-7}$ then the lemma holds since each of these edges commutes with edges in $E_{n-4}$. Additionally, if $S$ and $T$ are connected via edges labeled $2, 3, \ldots, n-4$, then we can assume $s$ and $t$ are also connected by edges in $E_2 \cup \ldots \cup E_{n-4}$ by Lemma 5.8. It thus suffices to show that some $S'$ in the same $\mathcal{G}^{[n-4]}$-component as $S$ and some $T'$ in the same $\mathcal{G}^{[n-4]}$-component as $T$ satisfy the following properties: both $S'$ and $T'$ admit $(n-3)$-edges that interchange $n-1$ and $n$, and both are in the same component of $\mathcal{C}^{[2,n-1]}$. By Lemma 5.7, it suffices to find $S'$ and $T'$ such that $\Delta(S')$ and $\Delta(T')$ have the same shape.

If $S|_{[n-2]}$ contains $i < n-3$ in a southeast boundary cell $c$ we can rearrange the entries of $S$ that are smaller than $i$ to obtain an $S'$ such that the cell $c$ is removed in $\Delta(S')$ and the rest of the jeu de taquin slides are independent of the filling. If $T|_{[n-2]}$ also contains an entry $j < n-3$ in cell $c$, we rearrange the entries of $T$ to agree with $S'$ in all cells weakly southwest of $c$ to obtain $T'$. Then $\Delta(T')$ moves the cell $c$ in the jeu de taquin process and by construction $\Delta(S')$ and $\Delta(T')$ have the same shape since $S$ and $T$ have the same shape, each with $n$ and $n-1$ in the same place. Thus by Lemma 5.7, $S'$ and $T'$ are connected by edges in $E_2 \cup \ldots \cup E_{n-4}$. If the shape of $S|_{[n-2]}$ has 5 or more southeast boundary cells, then such a cell $c$ exists and the lemma holds.

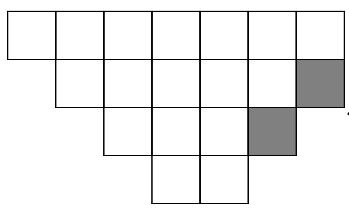
There are only a finite number of strict partitions $\mu$ with at most 4 southeast boundary cells after removing 2 corner cells. For example, if $\mu$ has 6 or more rows or 9 or more columns then even after removing 2 corner cells there must be at least

$$S = \begin{array}{cccccc} 1 & 2 & 3 & 6 & 7 & 9 \\ & 4 & 5 & & & \\ & & 8 & & & \end{array} \qquad T = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 9 \\ & 6 & 7 & & & \\ & & 8 & & & \end{array}$$

Figure 5.2: An example from the proof of Lemma 5.9 where $n = 9$ and $\Delta(S)$ does not have the same shape as $\Delta(T)$.

5 boundary cells remaining. We only need to consider such shapes with at least 10 cells by hypothesis. In each remaining case, one needs to check that no matter how $n$ and $n-1$ are placed in corner cells $\{c, d\}$ of $\mu$, the jeu de taquin argument above may still be applied. We leave the remaining cases to the reader to check to complete the proof. See Example 5.1 for an example of how such arguments progress. □

*Example* 5.1. Let $\lambda = (7, 6, 4, 2)$. If $n$ or $n-1$ is in the fourth row, there will be at least four southeast boundary cells remaining. Therefore, we see $n$ and $n-1$ must be in the second and third rows, as depicted below:



Since $n-2$ must be strictly between $n$ and $n-1$ in the reading word, it must be in the cell adjacent to $n$ and $n-1$. Additionally, $n-3$ must appear before these values in the reading word, so it must be in the first row. We can then arrange the jeu de taquin slides to pass through the last cell in the fourth row.

**Lemma 5.10.** *Let $\mathcal{G} = (V, \sigma, E_1 \cup \ldots \cup E_{n-3})$ be a connected signed colored graph of shifted degree $n > 9$ satisfying the Commuting Property such that $\mathcal{G}|_{[n-1]}$ is an SDEG, and $\mathcal{G}|_{[2,n]}$ is an SDEG. Then, there exists a morphism $\varphi \colon \mathcal{G} \to \mathcal{SG}_\lambda$ for some strict partition $\lambda \vdash n$.*

*Proof.* Let $u, v \in \mathcal{G}$ such that $u \sim_{n-3} v$. Let $\mathcal{C}$ and $\mathcal{D}$ be their components in $\mathcal{G}^{[n-1]}$ with unique extensions $\mathcal{SG}_\lambda$ and $\mathcal{SG}_\mu$, respectively. Label $u$ and $v$ with $T$ and $U$, their tableaux in $\mathcal{SG}_\lambda$ and $\mathcal{SG}_\mu$, respectively. It suffices to show that $h_{n-3}(T) = U$. In particular, this would guarantee that $\lambda = \mu$, and that there is a morphism from $\mathcal{G}$ to $\mathcal{SG}_\lambda$.

We first show that we can make three simplifying observations. If $T$ and $h_{n-3}(T)$ are in the same component of $\mathcal{SG}_\lambda^{[n-1]}$, then the result follows from Lemmas 5.9 and 5.1. Now assume that $T$ and $h_{n-3}(U)$ are in different components of $\mathcal{SG}_\lambda^{[n-1]}$. By symmetry, we may also assume that $U$ and $h_{n-3}(U)$ are in different components of $\mathcal{SG}_\mu^{[n-1]}$. Thus, we can assume $h_{n-3}$ acts on both $T$ and $U$ by switching $n-1$ and $n$.

Secondly, applying Lemma 5.1, the Commuting Property and the hypothesis that $\mathcal{G}|_{[n-1]}$ is an SDEG, it follows that $T|_{[n-4]} = U|_{[n-4]}$. We thus only need to show that $T|_{[n-3,n]} = h_{n-3}(U)|_{[n-3,n]}$.

For the third observation, note that the component of $T$ in $\mathcal{G}^{[2,n]}$ is isomorphic to the component $v$ in $\mathcal{SG}_\mu^{[2,n]}$. This follows by observing that the component of $T$ in $\mathcal{SG}_\lambda|_{[2,n]}$ is the unique extension of the component of $v$ in $\mathcal{G}|_{[2,n-1]}$ inside of $\mathcal{G}|_{[2,n]}$. Because the component of $v$ in $\mathcal{G}|_{[2,n]}$ is an SDEG, it follows from Lemma 5.1 that this component is isomorphic to the component of $T$ in $\mathcal{SG}_\lambda|_{[2,n]}$. Furthermore, $v$ is taken to $\Delta(T)$ via this isomorphism. Similarly, there is another isomorphism on the component of $w$ in $\mathcal{SG}^{[2,n]}$ taking $w$ to $\Delta(U)$. Applying Lemma 5.1 and the fact that $v$ and $w$ are in the same component of $\mathcal{SG}^{[2,n]}$, we see $\Delta(T) = h_{n-4} \circ \Delta(U)$.

By Lemma 5.9, we can instead consider some $T'$, obtained by rearranging the values $[1, n-2]$ in $T$, such that $h_{n-3}$ acts on $T'$ by switching $n-1$ and $n$. For each such $T'$, let $U'$ be the tableaux representing a vertex in $\mathcal{D}$ such that $(T', U') \in E_{n-3}$.

It suffices to show that $h_{n-3}(T') = U'$ for any collection of $(T', U')$ assuming that

(a) $T'|_{[n-4]} = U'|_{[n-4]}$.

(b) $\Delta(T') = h_{n-4}(\Delta(U'))$.

(c) $h_{n-3}$ acts on $T'$ by switching $n-1$ and $n$.

We proceed by considering cases depending on the shape of $T|_{[n-4]}$. First, consider the case where there is some $T'$ such that $T'|_{[n-4]}$ has at least two southeast corners $c_1$ and $c_2$. Assume $c_1$ is in a lower row than $c_2$. By rearranging the values in $[n-4]$, we may then construct $T_1'$ and $T_2'$ such that applying $\Delta$ to each proceeds through $c_1$ and $c_2$, respectively. In the jeu de taquin process on $T_1$, all rows strictly above $c_1$ are fixed once the slide reaches $c_1$. Similarly, all columns strictly to the left of $c_2$ are fixed once the slide reaches $c_2$. These two regions cover the entire shape of $T|_{[n-4]}$, but it might not cover the whole shape of $T$. By assumption (c), $n-1$ and $n$ must be in different corners of $T$. Thus, it can be observed that $T|_{[n-3,n]} = T_1|_{[n-3,n]} = T_2|_{[n-3,n]}$ is completely determined by their placement in $\Delta(T_1)$ and $\Delta(T_2)$. Similarly, $h_{n-3}(U)$ satisfies the same assumptions as $T$, which was uniquely determined, so $T = h_{n-3}(U)$.

Assume next that $T|_{[n-4]}$ has exactly one southeast corner $c$. In particular, the jeu de taquin process of applying $\Delta$ to $T$ must proceed through this corner. If $c$ is also a corner of both $\lambda$ and $\mu$, then the jeu de taquin process does not effect the cells containing $[n-3, n]$ in either $T$ or $U$ so $\Delta(T) = h_{n-4}(\Delta(U))$ implies $T = h_{n-3}(U)$.

Say $c$ is in row $i$ column $j$ of $T$. If $i > 3$ or $j-i > 3$, then there exists $c$ along the southeast boundary of both $T$ and $U$. Here we have used the fact that $h_{n-3}$ swaps $n-1$ and $n$ in $T$ and $U$ to ensure that the values in $[n-3, n]$ are not in a single row or column. Now consider the jeu de taquin process for $\Delta$, which must proceed through $c$. Since $c$ is on the southeast boundary, all remaining slides are either all to the left or all down depending only on whether or not $c$ is a southern boundary cell or an eastern boundary cell, respectively. Thus, $\Delta(T)$ determines $T|_{[n-3,n]}$ and $\Delta(U)$ determines $U|_{[n-3,n]}$. Hence, $T = h_{n-3}(U)$.

There are a finite number of standard shifted tableaux $T$ satisfying the assumptions such that $T|_{[n-4]}$ has a unique corner cell in position $(j, i)$ such that $i \leq 3$ and $j - i \leq 3$. We leave it to the reader to check in each case that if $T$ and $U$ exist satisfying the three assumptions plus they have at least 9 cells, then by rearranging the values $[n-2]$ one can find $T$ and $U$ of the same shape and satisfying the same assumptions such that $T|_{[n-3,n]}$ and $U|_{[n-3,n]}$ are completely determined by those assumptions and $T = h_{n-3}(U)$. This is discussed further in Example 5.2.

□

*Example* 5.2. To complete the proof of Lemma 5.10, we must verify the claim for shapes $\lambda \vdash n > 5$ with unique corner cell $(j, i)$ such that $i \leq 3$ and $j - i \leq 3$. These shapes are

$$\{(4,3), (5,4), (3,2,1), (4,3,2), (5,4,3), (6,5,4)\}.$$

We present an example for $\lambda = (4, 3)$. Let

$$S = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 9 & 11 \\ \hline \multicolumn{1}{c|}{} & 5 & 6 & 7 & 10 & \multicolumn{1}{c}{} \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} \qquad T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 9 & 11 \\ \hline \multicolumn{1}{c|}{} & 5 & 6 & 7 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & 10 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-4} \end{array} .$$

These tableau have the same shape after applying $\Delta$. We must verify that, by rearranging the values in $S$ and $T$, we can distinguish their shapes, given the assumption that satisfy (a), (b) and (c). To do so, we examine

$$S' = \begin{array}{cccccc} 1 & 2 & 3 & 7 & 9 & 11 \\ & 4 & 5 & 8 & 10 & \\ & & 6 & & & \end{array} \qquad T' = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 11 \\ & 6 & 7 & 9 & & \\ & & 8 & 10 & & \end{array}.$$

Note $\Delta(S')$ only has 2 rows, while $\Delta(T')$ has three, yet they satisfy our three assumptions. Therefore, we can distinguish the shapes of $S$ and $T$ by rearranging their entries and applying $\Delta$. More specifically, we can determine $S_{[8,11]}$ from $\Delta(S)$ and $\Delta(S')$, and similarly for $T_{[8,11]}$.

**Remark 5.1.** For $n = 8$, we may not be able to uniquely determine $U$ in the proof above. See Figure 5.3.

$$\text{T}= \begin{array}{cccc} 1 & 2 & 3 & 4 \\ & 5 & 6 & 7 \\ & & 8 & \end{array} \qquad \text{U}= \begin{array}{cccc} 1 & 2 & 3 & 4 \\ & 5 & 6 & 8 \\ & & 7 & \end{array}, \qquad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 8 \\ & 5 & 6 & & \\ & & 7 & & \end{array}$$

Figure 5.3: An example with $n = 8$ and two distinct possibilities of $U$ from the proof of Lemma 5.10.

The following lemma is equivalent to Axiom 6 in Assaf's rules for dual equivalence graphs.

**Lemma 5.11.** *Let $\mathcal{G} = (V, \sigma, E_1 \cup \ldots \cup E_{n-3})$ be a connected signed colored graph of shifted degree $n > 9$ satisfying the Commuting Property such that $\mathcal{G}|_{[n-1]}$ is an SDEG, and $\mathcal{G}|_{[2,n]}$ is an SDEG. Then each pair of distinct components of $\mathcal{G}^{[n-1]}$ is connected by an $(n-3)$-edge.*

This proof is similar to Theorem 3.17 in [34] and Lemma 5.10 so we only sketch it here.

*Proof.* Since $\mathcal{G}$ is connected, we can prove the lemma is equivalent to showing that if a component $\mathcal{A}$ is connected to components $\mathcal{B}$ and $\mathcal{C}$ by $(n-3)$-edges, then $\mathcal{B}$ and

$\mathcal{C}$ are connected to each other by an $(n-3)$-edge. Using Lemma 5.7, we may apply properties of jeu de taquin to show that this must be the case so long as $\lambda$ is not a pyramid or $\lambda$ has more than three southeast corners. The largest example of a shifted shape that violates these two rules is the pyramid $(5,3,1)$ with nine cells. By assumption, $n > 9$, and so the argument is complete. $\square$

**Proof of Theorem 5.1.** The fact that $\mathcal{SG}_\lambda$ satisfies the Commuting Property and the Locally Standard Property is proved in Lemma 5.2. To prove the converse, assume $\mathcal{G}$ is a signed colored graph with shifted degree $n$ satisfying both of these properties. We proceed by induction on $n$. For $n \leq 9$, the result holds by the Locally Standard Property. We may then assume $n > 9$.

By Lemma 5.10, $\mathcal{G}$ admits a morphism onto $\mathcal{SG}_\lambda$. By Lemma 5.5, we need only show that this morphism is injective. The morphism was constructed in such a way that it is the unique extension on any component of $\mathcal{G}^{[n-1]}$ so it is injective on each component automatically. Furthermore, the location of $n$ is constant on each component. Let $C$ and $D$ be two distinct components of $\mathcal{G}^{[n-1]}$, and let $v \in V(C)$ and $w \in V(D)$. By Lemma 5.11, there exists an $(n-3)$-edge connecting $C$ to $D$ which necessarily moves $n$ in the tableaux labeling its endpoints under the morphism. Thus, the morphism maps $v$ and $w$ to tableaux with $n$ in two different positions. Hence, the morphism is injective. $\square$

**Remark 5.2.** In Theorem 5.1, $n > 9$ is a sharp bound. In fact, if we consider the $n = 9$ case, then there exists an infinite family of such signed colored graphs that are not SDEGs, the smallest of which is represented in Figure 5.4.

$$
\begin{tikzcd}
 & \mathcal{SG}_{[5,3]} \arrow[r, dash, "6"] & \mathcal{SG}_{[5,2,1]} \arrow[dr, dash, "6"] & \\
\mathcal{SG}_{[4,3,1]} \arrow[ur, dash, "6"] \arrow[dr, dash, "6"'] & & & \mathcal{SG}_{[4,3,1]} \\
 & \mathcal{SG}_{[5,2,1]} \arrow[r, dash, "6"] & \mathcal{SG}_{[5,3]} \arrow[ur, dash, "6"'] &
\end{tikzcd}
$$

Figure 5.4: A graph $\mathcal{G}$ represented by the isomorphism types of components in $\mathcal{G}|_{[8]}$ and the connection of these components via 6-edges. Here, $\mathcal{G}$ satisfies the Commuting Property, $\mathcal{G}|_{[8]}$ and $\mathcal{G}|_{[2,9]}$ are SDEGs, but $\mathcal{G}$ is not an SDEG.

# Chapter 6

# Rothe diagrams of Types A and B

An important tool for tracking inversions in permutations is the *Rothe diagram*, introduced by Rothe in 1800. Young diagrams appear as special cases of Rothe diagrams. In this chapter we introduce Rothe diagrams and discuss balanced labelings. These labelings can be thought of as analogues of row and column strict labelings of tableaux.

In [10] Fomin, Greene, Reiner and Shimozono showed that for $w \in S_n$ the injective balanced labelings of the diagram of $w$ are in bijection with $R(w)$. This generalizes a result of Edelman and Greene that injective balanced labelings of tableaux are in bijection with standard tableaux. The results on balanced tableaux lead to a simple proof that the Tab map is a bijection. Following [10], we show how to define Type A Stanley symmetric functions as the generating functions of certain balanced labelings of the permutation diagram.

The final section of this chapter introduces a novel analog of Rothe diagrams for signed permutations. For $w \in B_n$, we show that injective balanced labelings of its diagram are in bijection with $R(w)$.

## 6.1 Rothe diagrams

We begin by introducing Rothe diagrams for permutations.

**Definition 6.1.** For $w \in S_\infty$, the *Rothe diagram* of $w$ is the set

$$D(w) = \{(w_j, i) : i < j, w_i > w_j\} \subset \mathbb{N} \times \mathbb{N}.$$

Note $|D(w)| = |\mathrm{Inv}(w)|$, since the entries of $D(w)$ are indexed by inversions. There is a simple way to construct $D(w)$. For a permutation $w \in S_n$, place an $X$ in the entries of the $n \times n$ diagram with labels $(w_i, -i)$. Place $\cdot$ in each entry below or to the right of an $X$. The remaining entries will be $D(w)$, which are denoted by $\bigcirc$. Below is the diagram of the permutation $w = 35214$.

| | 3 | 5 | 2 | 1 | 4 |
|---|---|---|---|---|---|
| 1 | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $X$ | $\cdot$ |
| 2 | $\bigcirc$ | $\bigcirc$ | $X$ | $\cdot$ | $\cdot$ |
| 3 | $X$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| 4 | $\cdot$ | $\bigcirc$ | $\cdot$ | $\cdot$ | $X$ |
| 5 | $\cdot$ | $X$ | $\cdot$ | $\cdot$ | $\cdot$ |

Many properties of a permutation can be deduced from its diagram. We present some examples, as seen in [30]. For example, a permutation is vexillary (2143-avoiding) if an only if the rows of its diagram are totally ordered by inclusion. Equivalently, the columns of the diagram are totally ordered by inclusion. The diagrams of Grassmannian permutations have the additional feature that the ordering for non-empty rows is from bottom to top, while the ordering for non-empty columns is from left to right. This can be seen in the diagram of the Grassmannian permutation $w = 3671245$.

| | 3 | 6 | 7 | 1 | 2 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| 1 | ◯ | ◯ | ◯ | $X$ | · | · | · |
| 2 | ◯ | ◯ | ◯ | · | $X$ | · | · |
| 3 | $X$ | · | · | · | · | · | · |
| 4 | · | ◯ | ◯ | · | · | $X$ | · |
| 5 | · | ◯ | ◯ | · | · | · | $X$ |
| 6 | · | $X$ | · | · | · | · | · |
| 7 | · | · | $X$ | · | · | · | · |

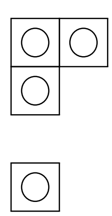
The columns of this diagram form the partition $(4, 4, 2)$. Billey, Jockusch and Stanley showed that diagrams of 321-avoiding permutations correspond to skew-shapes in [7]. Permutations whose diagrams are Young diagrams are called *dominant*.

## 6.2 Balanced labelings

We can view permutation diagrams as generalizing Young diagrams. From this perspective, balanced labelings are the correct generalization of the row and column strict conditions of Young tableaux. For a diagram $D$ containing $(i, j)$, the *arm* and *leg* of $(i, j)$ are $A_{(i,j)}$ and $L_{(i,j)}$ respectively, where

$$A_{(i,j)} = \{(k, j) \in D : k > i\} \quad \text{and} \quad L_{(i,j)} = \{(i, l) : l < j\}.$$

The *hook* of $(i, j)$ is $H_{(i,j)} = \{(i, j)\} \cup A_{(i,j)} \cup L_{(i,j)}$. For example, in the diagram of 35214, the hook of $(2, -1)$, indexed by $(1, 5)$, is



**Definition 6.2.** Let $D$ be a diagram with labeling $s$ and $p$ cells, including the cell $(i,j)$. Let $l^+_{(i,j)}$ be the number of cells in the leg $L_{(i,j)}$ with labels less than or equal to that of $(i,j)$, and $l^-_{(i,j)}$ be the number of such cells whose label is strictly less than that of $(i,j)$. Define $a^+_{(i,j)}$ and $a^-_{(i,j)}$ similarly, with respect to the arm $A_{(i,j)}$. Implicitly, we require a total ordering on the labels. Note the ordering used to determine the $l$'s need not be the same as the ordering used to determine the $a$'s.

We say the hook $H_(i,j)$ is *balanced* with respect to $s$ if $|L_{(i,j)}| \in [a^-_{(i,j)}+l^-_{(i,j)}, a^+_{(i,j)}+l^+_{(i,j)}]$. When the labels are distinct, this is equivalent to saying the number of entries in the leg is the same as the number of entries in the hook less than $(i,j)$'s label. The labeling $s$ is *balanced* if every hook in $D$ is balanced with respect to $s$. Moreover, $s$ is an *injective balanced labeling* if its image is $[p]$. We denote the set of injective balanced labelings of $D$ by $B(D)$.

The balanced condition appears naturally in the setting of reduced words. Let $w \in S_\infty$ and $a \in R(w)$. We define the labeled diagram $D_a(w)$ by assigning to the entry indexed by $(i,j)$ the time $k$ when the $i$ and $j$ trajectories cross. In order for trajectories $i$ and $j$ with $i < j$ to cross, they must first move next to each other. For this to happen, the $j-i-1$ trajectories between $i$ and $j$ must first cross either $i$ or $j$. Each of these crossings will appear in the hook of the entry indexed by $(i,j)$. The entries in $L_{(i,j)}$ index these swaps. Therefore, $D_a(w)$ is balanced.

Below is the balanced labeling $D_a(w)$ for $a = 24312545341 \in R(536412)$.

| | 5 | 3 | 6 | 4 | 1 | 2 |
|---|---|---|---|---|---|---|
| 1 | 5 | 4 | 9 | 10 | $X$ | $\cdot$ |
| 2 | 3 | 1 | 7 | 8 | $\cdot$ | $X$ |
| 3 | 11 | $X$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| 4 | 2 | $\cdot$ | 6 | $X$ | $\cdot$ | $\cdot$ |
| 5 | $X$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| 6 | $\cdot$ | $\cdot$ | $X$ | $\cdot$ | $\cdot$ | $\cdot$ |

Fomin, Greene, Reiner and Shimozono proved that all injective balanced labelings of Rothe diagrams arise from reduced words.

**Theorem 6.1** ([10], Theorem 2.4)**.** *Let $w \in S_\infty$ and $a \in R(w)$. The map $a \mapsto D_a(w)$ is a bijection between $R(w)$ and $B(D(w))$.*

As a corollary, since every Young diagram is also the Rothe diagram of a vexillary permutation, we can recover an earlier result of Edelman and Greene.

**Theorem 6.2** ([9], Theorem 2.2)**.** *Let $\lambda$ be a partition. Then*

$$|B(\lambda)| = |SYT(\lambda)|.$$

Given an injective balanced of the Rothe diagram $D(w)$, we construct the corresponding reduced word by removing entries one at a time and using their positions to build the word. Identify the entry $(i, j)$ with the largest label in $D$. Prepend $i$ to the beginning of the word $a$. Swap columns $i$ and $i + 1$. This is equivalent to applying $s_i$ to $w$ and will remove the entry $(i, j)$ from $D(w)$. Repeat until there are no more entries in $D(w)$.

We present an example of this algorithm.

$$\begin{array}{c|c|c|c|c|c|} & 3 & 5 & 2 & 1 & 4 \\ \cline{2-6} 1 & 3 & 5 & 2 & X & \cdot \\ \cline{2-6} 2 & 4 & 6 & X & \cdot & \cdot \\ \cline{2-6} 3 & X & \cdot & \cdot & \cdot & \cdot \\ \cline{2-6} 4 & \cdot & 1 & \cdot & \cdot & X \\ \cline{2-6} 5 & \cdot & X & \cdot & \cdot & \cdot \\ \cline{2-6} \end{array} \qquad \begin{array}{c|c|c|c|c|c|} & 3 & 2 & 5 & 1 & 4 \\ \cline{2-6} 1 & 3 & 2 & 5 & X & \cdot \\ \cline{2-6} 2 & 4 & X & \cdot & \cdot & \cdot \\ \cline{2-6} 3 & X & \cdot & \cdot & \cdot & \cdot \\ \cline{2-6} 4 & \cdot & \cdot & 1 & \cdot & X \\ \cline{2-6} 5 & \cdot & \cdot & X & \cdot & \cdot \\ \cline{2-6} \end{array}$$

$a = 2$ $\qquad$ $a = 32$

$$\begin{array}{c|c|c|c|c|c|} & 3 & 2 & 1 & 5 & 4 \\ \cline{2-6} 1 & 3 & 2 & X & \cdot & \cdot \\ \cline{2-6} 2 & 4 & X & \cdot & \cdot & \cdot \\ \cline{2-6} 3 & X & \cdot & \cdot & \cdot & \cdot \\ \cline{2-6} 4 & \cdot & \cdot & \cdot & 1 & X \\ \cline{2-6} 5 & \cdot & \cdot & \cdot & X & \cdot \\ \cline{2-6} \end{array} \qquad \begin{array}{c|c|c|c|c|c|} & 2 & 3 & 1 & 5 & 4 \\ \cline{2-6} 1 & 2 & 3 & X & \cdot & \cdot \\ \cline{2-6} 2 & X & \cdot & \cdot & \cdot & \cdot \\ \cline{2-6} 3 & \cdot & X & \cdot & \cdot & \cdot \\ \cline{2-6} 4 & \cdot & \cdot & \cdot & 1 & X \\ \cline{2-6} 5 & \cdot & \cdot & \cdot & X & \cdot \\ \cline{2-6} \end{array}$$

$a = 132$ $\qquad$ $a = 2132$

$$\begin{array}{c|c|c|c|c|c|} & 2 & 1 & 3 & 5 & 4 \\ \cline{2-6} 1 & 2 & X & \cdot & \cdot & \cdot \\ \cline{2-6} 2 & X & \cdot & \cdot & \cdot & \cdot \\ \cline{2-6} 3 & \cdot & \cdot & X & \cdot & \cdot \\ \cline{2-6} 4 & \cdot & \cdot & \cdot & 1 & X \\ \cline{2-6} 5 & \cdot & \cdot & \cdot & X & \cdot \\ \cline{2-6} \end{array} \qquad \begin{array}{c|c|c|c|c|c|} & 1 & 2 & 3 & 5 & 4 \\ \cline{2-6} 1 & X & \cdot & \cdot & \cdot & \cdot \\ \cline{2-6} 2 & \cdot & X & \cdot & \cdot & \cdot \\ \cline{2-6} 3 & \cdot & \cdot & X & \cdot & \cdot \\ \cline{2-6} 4 & \cdot & \cdot & \cdot & 1 & X \\ \cline{2-6} 5 & \cdot & \cdot & \cdot & X & \cdot \\ \cline{2-6} \end{array}$$

$a = 12132$ $\qquad$ $a = 412132$

An alternative description of the map from reduced words to injective balanced diagrams is demonstrated in [21]. To a number $k$ we associate the matrix $M_k$, which is the permutation matrix of $s_k$ with $x_k$ in the $(k, k)$th entry. For $w \in S_\infty$ and $a = a_1 a_2 \dots a_p \in R(w)$, let $M_a = M_{a_1} M_{a_2} \dots M_{a_p}$. We can decompose $M_a$ as the permutation matrix of $w$ together with linear, quadratic and higher order terms. After applying the map $x_i \mapsto i$, the linear term is $D_a(w)$. The non-linear component encodes additional information.

**Theorem 6.3** ([21], Theorem 1.3)**.** *Let $w \in S_\infty$ and $a, b \in R(w)$. The words $a$ can be* *obtained from b by a sequence of commuting Coxeter moves if and only if they have the same quadratic term.*

Recall for $a = a_1 \dots a_p$ a reduced word that $K(a)$ are the compatible sequences of the form $i_1 i_2 \dots i_p$ where $i_k = i_{k+1}$ implies $a_k > a_{k+1}$. Note that the fundamental quasisymmetric function associated to $a$ is the generating function of $K(a)$, that is

$$f_{\sigma_a} = \sum_{i_1 \dots i_p \in K(a)} x_{i_1} \dots x_{i_p}.$$

Given a compatible sequence $I = i_1 \dots i_p \in K(a)$, we obtain a labeled diagram $D_I(w)$ by replacing $a_k$ with $i_k$ for all labels in $D_a(w)$. Such diagrams are clearly balanced. Moreover, they are row strict. Let $BR(D)$ be the set of balanced, row strict labelings of the diagram $D$. Fomin, Greene, Reiner and Shimozono demonstrated that every member of $BR(D(w))$ arises in this way, proving the following theorem.

**Theorem 6.4** ([10], Theorem 4.3)**.** *Let $w \in S_\infty$. Then*

$$F_w = \sum_{D \in BR(D(w))} x^D$$

*where $x^D = x^{i_1} \dots x^{i_p}$ where $i_1, \dots, i_p$ are the labels of entries in $D$.*

## 6.3 Type B Rothe diagrams

We now develop an analogue of Rothe diagrams for signed permutations. Let $w = w_{\bar{n}} \dots w_{\bar{1}} w_1 \dots w_n \in B_\infty$. The *Type B Rothe diagram* of $w$ is

$$D'(w) = \{(w_j, i) : i < j, w_i > w_j\} / \{(w_j, i) \sim (w_{\bar{i}}, j)\}$$

Note this is the Type A diagram of $\mathrm{fl}(w)$ with the entries $(w_j, i)$ and $(w_{\bar{i}}, -j)$ identified. In particular, a labeling of $D'(w)$ will give the same value to both these entries. One depiction of $D'(w)$ is quite similar to that of $D(\mathrm{fl}(w))$, and we show $D'(3\bar{1}\bar{4}2)$ below.

$$
\begin{array}{r|c|c|c|c|c|c|c|c|}
 & -2 & 1 & 4 & -3 & 3 & -4 & -1 & 2 \\
\hline
-4 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & X & \cdot & \cdot \\
\hline
-3 & \bigcirc & \bigcirc & \bigcirc & X & \cdot & \cdot & \cdot & \cdot \\
\hline
-2 & X & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
-1 & \cdot & \bigcirc & \bigcirc & \cdot & \bigcirc & \cdot & X & \cdot \\
\hline
1 & \cdot & X & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
2 & \cdot & \cdot & \bigcirc & \cdot & \bigcirc & \cdot & \cdot & X \\
\hline
3 & \cdot & \cdot & \bigcirc & \cdot & X & \cdot & \cdot & \cdot \\
\hline
4 & \cdot & \cdot & X & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}
$$

The Type B Rothe diagram of a signed permutation $w$ can be used in a similar way as the Type A Rothe diagram to characterize properties of $w$. The flatten applied to vexillary and isotropic Grassmannian signed permutations produces permutations that are vexillary and Grassmannian, respectively. Therefore, their diagrams inherit the properties of vexillary and Grassmannian Rothe diagrams.

**Lemma 6.1.** *Let $w = w_{\bar{n}} \dots w_{\overline{(1)}} w_1 \dots w_n$ be an isotropic Grassmannian. Then the partition associated to the diagram of $\mathit{fl}(w)$ is self-conjugate, with entries across the diagonal identified.*

*Proof.* The shape associated to $D(\mathrm{fl}(w))$ is $\lambda = \lambda_1 \dots \lambda_n$ where

$$\lambda_i = |\{j \in [-n] : w_j > w_i\}|.$$

Since $w_{\bar{n}} \dots w_{\bar{1}}$ and $w_1 \dots w_n$ are increasing, we see

$$\lambda'_i = |\{j \in [n] : w_j < w_{\bar{i}}\}|.$$

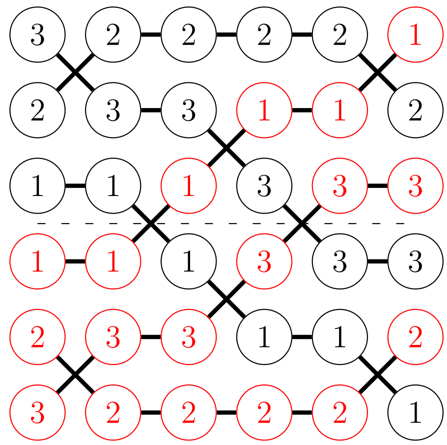
Since $w_i = -w_{\bar{i}}$, we see $\lambda_i = \lambda'_i$ for all $i$.

$\square$

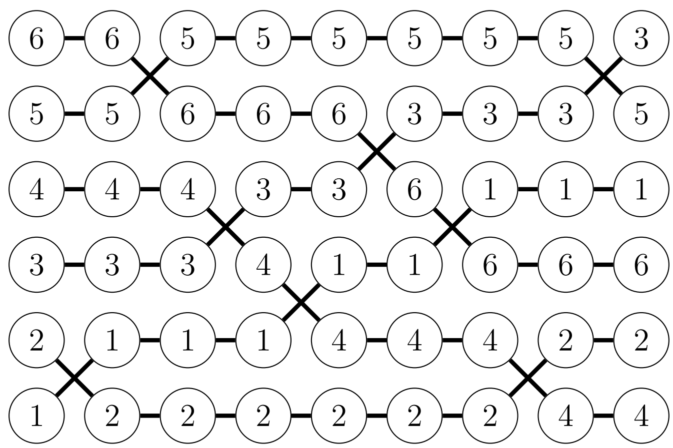
By Lemma 6.1, we can associate to each isotropic Grassmannian $w$ the shape $\lambda_w$ obtained by restricting $\mathrm{fl}(w)$ to a shifted shape. Below is the diagram of the isotropic Grassmannian permutation $w = \bar{3}\bar{2}14\bar{4}\bar{1}23$, with shifted shape $\lambda = (4, 1)$.

| | $-3$ | $-2$ | $1$ | $4$ | $-4$ | $-1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|---|
| $-4$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $X$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $-3$ | $X$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $-2$ | $\cdot$ | $X$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $-1$ | $\cdot$ | $\cdot$ | $\bigcirc$ | $\bigcirc$ | $\cdot$ | $X$ | $\cdot$ | $\cdot$ |
| $1$ | $\cdot$ | $\cdot$ | $X$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $2$ | $\cdot$ | $\cdot$ | $\cdot$ | $\bigcirc$ | $\cdot$ | $\cdot$ | $X$ | $\cdot$ |
| $3$ | $\cdot$ | $\cdot$ | $\cdot$ | $\bigcirc$ | $\cdot$ | $\cdot$ | $\cdot$ | $X$ |
| $4$ | $\cdot$ | $\cdot$ | $\cdot$ | $X$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |

Balanced labelings of Type B Rothe diagrams also correspond to reduced words of signed permutations. To show this, we will view reduced words of signed permutations as reduced words of permutations. Let $w \in B_{n-1}$ and $a = a_1 \dots a_p \in R(w)$. By mapping $a_i$ to $(n - a_i)(n + a_i)$ if $a_i > 0$ and to $n$ if $a_i = 0$, we obtain a reduced word $b_{\mathrm{fl}}(a) = b_1 \dots b_{2p-l_0(w)} \in R(\mathrm{fl}(w))$. Recall $l_0(w)$ is the number of zeros in a reduced word of $w$. For example, the reduced word $20102 \in R(\bar{3}21)$ corresponds to $b_{\mathrm{fl}}(20102) = 1532415 \in R(426153)$.



As an action on wiring diagrams, $b_{\text{fl}}$ acts by separating the crossings in the same column into separate columns.



Since $n + a_i - (n - a_i) = 2a_i > 1$ for $a_i \neq 0$, the split entries commute via Coxeter moves. The choice of splitting is not canonical.

For $w \in B_\infty$ and $a \in R(w)$, we define $D'_a(w)$ by assigning to the entry indexed by $(i, j)$ the time $k$ when the $i$ and $j$ entries cross. Note this labeling is injective, since each entry has a unique value, up to equivalence. Since $i = -j$ implies the inversion $(i, j) = (i, -i)$ is introduced by $s_0$, we see two entries with the same label must be in different rows and different columns. This implies they are in different hooks. Therefore, hooks in $D_{b_{\text{fl}}(a)}(\text{fl}(w))$ and $D'_a(w)$ have they same relative order, so $D'_a(w)$ is balanced. Extending this analogy, we can prove an analogue of Theorem for Type B Rothe diagrams. Let $B(D'(w))$ be the set of injective balanced labelings of $D'(w)$.

**Theorem 6.5.** *Let $w \in B_\infty$ and $a \in R(w)$. The map $a = a_1 \ldots a_p \mapsto D'_a(w)$ is a bijection between $R(w)$ and $B(D'(w))$.*

*Proof.* We have already shown that $D'_a(w) \in B(D'(w))$. Let $C \in B(D'(w))$. We construct some labeling $A_{\text{fl}} \in B(D(\text{fl}(w)))$ by giving each identified pair of entries in $A$ different labels while otherwise preserving the relative order labels in $C$. This can be done in many ways, but the particular construction does not matter. For example,

$$\begin{array}{c|c|c|c|c|c|c|}
 & -1 & 2 & -3 & 3 & -2 & 1 \\
\hline
-3 & 2 & 4 & X & \cdot & \cdot & \cdot \\
\hline
-2 & 1 & 3 & \cdot & 4 & X & \cdot \\
\hline
-1 & X & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
1 & \cdot & 1 & \cdot & 2 & \cdot & X \\
\hline
2 & \cdot & X & \cdot & \cdot & \cdot & \cdot \\
\hline
3 & \cdot & \cdot & \cdot & X & \cdot & \cdot \\
\hline
\end{array}
\qquad
\begin{array}{c|c|c|c|c|c|c|}
 & 3 & 5 & 1 & 6 & 2 & 4 \\
\hline
1 & 3 & 6 & X & \cdot & \cdot & \cdot \\
\hline
2 & 2 & 5 & \cdot & 7 & X & \cdot \\
\hline
3 & X & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
4 & \cdot & 1 & \cdot & 4 & \cdot & X \\
\hline
5 & \cdot & X & \cdot & \cdot & \cdot & \cdot \\
\hline
6 & \cdot & \cdot & \cdot & X & \cdot & \cdot \\
\hline
\end{array}.$$

Since $C$ is balanced, it corresponds to some reduced word $c \in R\text{fl}(w))$ via the inverse map of $c \mapsto D_c(w)$. As we apply this map, we will simultaneously produce a word $a$. At the first step of the inverse map, we deduce the last entry in $c$. If this entry is $n$, we add 0 to $a$. If this entry is $j \neq n$, the second to last entry in $c$ will be $j + 2(n-j)$, since the corresponding entries in $C$ were identified. In this case, we add $|n-j|$ to $a$. In our example above, we see the last entry of $c$ is 4, followed by 2 so the last entry of $a$ will 1. Continuing this process, we obtain $a$.

To see $a \in R(w)$, we first observe that $b_{\text{fl}}(a) \in R(\text{fl}(w))$ since it differs from $c$ by commuting Coxeter relations. Therefore, $a$ must be an expression of $w$. Moreover, it is reduced because $l(a) = l(w)$.

$\square$

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